Chapter Five

Similarity

We have shown that for any homomorphism there are bases B and D such that the matrix representing the map has a block partial-identity form.

$$\mathrm{Rep}_{B,D}(h)=\left(\begin{array}{c|c} \textit{Identity} & \textit{Zero} \\ \hline \textit{Zero} & \textit{Zero} \end{array}\right)$$

This representation describes the map as sending $c_1\vec{\beta}_1+\cdots+c_n\vec{\beta}_n$ to $c_1\vec{\delta}_1+\cdots+c_k\vec{\delta}_k+\vec{0}+\cdots+\vec{0}$, where n is the dimension of the domain and k is the dimension of the range. Under this representation the action of the map is easy to understand because most of the matrix entries are zero.

This chapter considers the special case where the domain and codomain are the same. Here we naturally ask for the domain basis and codomain basis to be the same. That is, we want a basis B so that $\mathrm{Rep}_{B,B}(t)$ is as simple as possible, where we take 'simple' to mean that it has many zeroes. We will find that we cannot always get a matrix having the above block partial-identity form but we will develop a form that comes close, a representation that is nearly diagonal.

I Complex Vector Spaces

This chapter requires that we factor polynomials. But many polynomials do not factor over the real numbers; for instance, x^2+1 does not factor into a product of two linear polynomials with real coefficients, instead it requires complex numbers $x^2+1=(x-i)(x+i)$.

Consequently in this chapter we shall use complex numbers for our scalars, including entries in vectors and matrices. That is, we shift from studying vector spaces over the real numbers to vector spaces over the complex numbers. Any real number is a complex number and in this chapter most of the examples use only real numbers but nonetheless, the critical theorems require that the scalars be complex. So this first section is a review of complex numbers.

In this book our approach is to shift to this more general context of taking scalars to be complex for the pragmatic reason that we must do so in order to move forward. However, the idea of doing vector spaces by taking scalars from a structure other than the real numbers is an interesting and useful one. Delightful presentations that take this approach from the start are in [Halmos] and [Hoffman & Kunze].

I.1 Polynomial Factoring and Complex Numbers

This subsection is a review only. For a full development, including proofs, see [Ebbinghaus].

Consider a polynomial $p(x) = c_n x^n + \cdots + c_1 x + c_0$ with leading coefficient $c_n \neq 0$ and $n \geqslant 1$. The degree of the polynomial is n. If $n = 0$ then p is a constant polynomial $p(x) = c_0$. Constant polynomials that are not the zero polynomial, $c_0 \neq 0$, have degree zero. We define the zero polynomial to have degree $-\infty$.

1.1 Remark Defining the degree of the zero polynomial to be $-\infty$ allows the equation $\text{degree}(fg) = \text{degree}(f) + \text{degree}(g)$ to hold for all polynomials.

Just as integers have a division operation — e.g., '4 goes 5 times into 21 with remainder 1' — so do polynomials.

1.2 Theorem (Division Theorem for Polynomials) Let $p(x)$ be a polynomial. If $d(x)$ is a non-zero polynomial then there are *quotient* and *remainder* polynomials $q(x)$ and $r(x)$ such that

$$p(x) = d(x) \cdot q(x) + r(x)$$

where the degree of $r(x)$ is strictly less than the degree of $d(x)$.

The point of the integer statement '4 goes 5 times into 21 with remainder 1' is that the remainder is less than 4 — while 4 goes 5 times, it does not go 6 times. Similarly, the final clause of the polynomial division statement is crucial.

1.3 Example If $p(x) = 2x^3 - 3x^2 + 4x$ and $d(x) = x^2 + 1$ then $q(x) = 2x - 3$ and $r(x) = 2x + 3$. Note that $r(x)$ has a lower degree than does $d(x)$.

1.4 Corollary The remainder when $p(x)$ is divided by $x - \lambda$ is the constant polynomial $r(x) = p(\lambda)$.

Proof The remainder must be a constant polynomial because it is of degree less than the divisor $x - \lambda$. To determine the constant, take the theorem's divisor $d(x)$ to be $x - \lambda$ and substitute λ for x. QED

If a divisor $d(x)$ goes into a dividend $p(x)$ evenly, meaning that $r(x)$ is the zero polynomial, then $d(x)$ is a called a *factor* of $p(x)$. Any *root* of the factor, any $\lambda \in \mathbb{R}$ such that $d(\lambda) = 0$, is a root of $p(x)$ since $p(\lambda) = d(\lambda) \cdot q(\lambda) = 0$.

1.5 Corollary If λ is a root of the polynomial $p(x)$ then $x - \lambda$ divides $p(x)$ evenly, that is, $x - \lambda$ is a factor of $p(x)$.

Proof By the above corollary $p(x) = (x - \lambda) \cdot q(x) + p(\lambda)$. Since λ is a root, $p(\lambda) = 0$ so $x - \lambda$ is a factor. QED

A *repeated* root of a polynomial is a number λ such that the polynomial is evenly divisible by $(x - \lambda)^n$ for some power larger than one. The largest such power is called the *multiplicity* of λ.

Finding the roots and factors of a high-degree polynomial can be hard. But for second-degree polynomials we have the quadratic formula: the roots of $ax^2 + bx + c$ are these

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(if the discriminant $b^2 - 4ac$ is negative then the polynomial has no real number roots). A polynomial that cannot be factored into two lower-degree polynomials with real number coefficients is said to be *irreducible over the reals*.

1.6 Theorem Any constant or linear polynomial is irreducible over the reals. A quadratic polynomial is irreducible over the reals if and only if its discriminant is negative. No cubic or higher-degree polynomial is irreducible over the reals.

1.7 Corollary Any polynomial with real coefficients can be factored into linear and irreducible quadratic polynomials. That factorization is unique; any two factorizations have the same powers of the same factors.

Note the analogy with the prime factorization of integers. In both cases the uniqueness clause is very useful.

1.8 Example Because of uniqueness we know, without multiplying them out, that $(x+3)^2(x^2+1)^3$ does not equal $(x+3)^4(x^2+x+1)^2$.

1.9 Example By uniqueness, if $c(x) = m(x) \cdot q(x)$ then where $c(x) = (x-3)^2(x+2)^3$ and $m(x) = (x-3)(x+2)^2$, we know that $q(x) = (x-3)(x+2)$.

While x^2+1 has no real roots and so doesn't factor over the real numbers, if we imagine a root — traditionally denoted i, so that $i^2+1=0$ — then x^2+1 factors into a product of linears $(x-i)(x+i)$. When we adjoin this root i to the reals and close the new system with respect to addition and multiplication then we have the complex numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R} \text{ and } i^2 = 1\}$. (These are often pictured on a plane with a plotted on the horizontal axis and b on the vertical; note that the distance of the point from the origin is $|a+bi| = \sqrt{a^2+b^2}$.)

In $\mathbb{C}$ all quadratics factor. That is, in contrast with the reals, $\mathbb{C}$ has no irreducible quadratics.

$$ax^2 + bx + c = a \cdot \big(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\big) \cdot \big(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\big)$$

1.10 Example The second degree polynomial $x^2 + x + 1$ factors over the complex numbers into the product of two first degree polynomials.

$$\big(x - \frac{-1 + \sqrt{-3}}{2}\big)\big(x - \frac{-1 - \sqrt{-3}}{2}\big) = \big(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2}i)\big)\big(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2}i)\big)$$

1.11 Theorem (Fundamental Theorem of Algebra) Polynomials with complex coefficients factor into linear polynomials with complex coefficients. The factorization is unique.

I.2 Complex Representations

Recall the definitions of the complex number addition

$$(a+bi) \;+\; (c+di) = (a+c) + (b+d)i$$

and multiplication.

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bd(-1) \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

2.1 Example For instance, $(1-2i) \;+\; (5+4i) = 6+2i$ and $(2-3i)(4-0.5i) = 6.5 - 13i$.

With these rules, all of the operations that we've used for real vector spaces carry over unchanged to vector spaces with complex scalars.

2.2 Example Matrix multiplication is the same, although the scalar arithmetic involves more bookkeeping.

$$\begin{aligned}
&\begin{pmatrix} 1+1i & 2-0i \\ i & -2+3i \end{pmatrix} \begin{pmatrix} 1+0i & 1-0i \\ 3i & -i \end{pmatrix} \\
&= \begin{pmatrix} (1+1i)\cdot(1+0i)+(2-0i)\cdot(3i) & (1+1i)\cdot(1-0i)+(2-0i)\cdot(-i) \\ (i)\cdot(1+0i)+(-2+3i)\cdot(3i) & (i)\cdot(1-0i)+(-2+3i)\cdot(-i) \end{pmatrix} \\
&= \begin{pmatrix} 1+7i & 1-1i \\ -9-5i & 3+3i \end{pmatrix}
\end{aligned}$$

We shall carry over unchanged from the previous chapters everything that we can. For instance, we shall call this

$$\langle \begin{pmatrix} 1+0i \\ 0+0i \\ \vdots \\ 0+0i \end{pmatrix}, \dots, \begin{pmatrix} 0+0i \\ 0+0i \\ \vdots \\ 1+0i \end{pmatrix} \rangle$$

the *standard basis* for $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ and again denote it $\mathcal{E}_n$.

II Similarity

We've defined two matrices H and $\hat{H}$ to be matrix equivalent if there are nonsingular P and Q such that $\hat{H} = PHQ$. We were motivated by this diagram showing H and $\hat{H}$ both representing a map h, but with respect to different pairs of bases, B, D and $\hat{B}, \hat{D}$.

$$\begin{array}{ccc} V_{\mathit{wrt}\ B} & \xrightarrow[H]{h} & W_{\mathit{wrt}\ D} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\mathit{wrt}\ \hat{B}} & \xrightarrow[\hat{H}]{h} & W_{\mathit{wrt}\ \hat{D}} \end{array}$$

We now consider the special case of transformations, where the codomain equals the domain, and we add the requirement that the codomain's basis equals the domain's basis. So, we are considering representations with respect to B, B and D, D.

$$\begin{array}{ccc} V_{\mathit{wrt}\ B} & \xrightarrow[T]{t} & V_{\mathit{wrt}\ B} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\mathit{wrt}\ D} & \xrightarrow[\hat{T}]{t} & V_{\mathit{wrt}\ D} \end{array}$$

In matrix terms, $\mathrm{Rep}_{D,D}(t) = \mathrm{Rep}_{B,D}(\mathrm{id})\ \mathrm{Rep}_{B,B}(t)\ \big(\mathrm{Rep}_{B,D}(\mathrm{id})\big)^{-1}$.

II.1 Definition and Examples

1.1 Definition The matrices T and $\hat{T}$ are *similar* if there is a nonsingular P such that $\hat{T} = PTP^{-1}$.

Since nonsingular matrices are square, T and $\hat{T}$ must be square and of the same size. Exercise 12 checks that similarity is an equivalence relation.

1.2 Example Calculation with these two

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$$

gives that T is similar to this matrix.

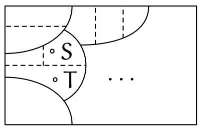
$$\hat{T} = \begin{pmatrix} 12 & -19 \\ 7 & -11 \end{pmatrix}$$

1.3 Example The only matrix similar to the zero matrix is itself: $PZP^{-1} = PZ = Z$. The identity matrix has the same property: $PIP^{-1} = PP^{-1} = I$.

Matrix similarity is a special case of matrix equivalence so if two matrices are similar then they are matrix equivalent. What about the converse: if they are square, must any two matrix equivalent matrices be similar? No; the matrix equivalence class of an identity matrix consists of all nonsingular matrices of that size while the prior example shows that the only member of the similarity class of an identity matrix is itself. Thus these two are matrix equivalent but not similar.

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

So some matrix equivalence classes split into two or more similarity classes — similarity gives a finer partition than does matrix equivalence. This shows some matrix equivalence classes subdivided into similarity classes.



To understand the similarity relation we shall study the similarity classes. We approach this question in the same way that we've studied both the row equivalence and matrix equivalence relations, by finding a canonical form for representatives of the similarity classes, called Jordan form. With this canonical form, we can decide if two matrices are similar by checking whether they are in a class with the same representative. We've also seen with both row equivalence and matrix equivalence that a canonical form gives us insight into the ways in which members of the same class are alike (e.g., two identically-sized matrices are matrix equivalent if and only if they have the same rank).

Exercises

1.4 For

$$T = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix} \quad \hat{T} = \begin{pmatrix} 0 & 0 \\ -11/2 & -5 \end{pmatrix} \quad P = \begin{pmatrix} 4 & 2 \\ -3 & 2 \end{pmatrix}$$

check that $\hat{T} = PTP^{-1}$.

✓ 1.5 Example 1.3 shows that the only matrix similar to a zero matrix is itself and that the only matrix similar to the identity is itself.

(a) Show that the 1×1 matrix whose single entry is 2 is also similar only to itself.

(b) Is a matrix of the form cI for some scalar c similar only to itself?

(c) Is a diagonal matrix similar only to itself?

✓ 1.6 Show that these matrices are not similar.

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & 1 & 7 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

1.7 Consider the transformation $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ described by $x^2 \mapsto x+1$, $x \mapsto x^2-1$, and $1 \mapsto 3$.

(a) Find $T = \mathrm{Rep}_{B,B}(t)$ where $B = \langle x^2, x, 1\rangle$.

(b) Find $\hat{T} = \mathrm{Rep}_{D,D}(t)$ where $D = \langle 1, 1+x, 1+x+x^2\rangle$.

(c) Find the matrix P such that $\hat{T} = PTP^{-1}$.

✓ 1.8 Exhibit an nontrivial similarity relationship in this way: let $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ act in this way,

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \mapsto \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

and pick two bases, and represent t with respect to them $\hat{T} = \mathrm{Rep}_{B,B}(t)$ and $T = \mathrm{Rep}_{D,D}(t)$. Then compute the P and P^{-1} to change bases from B to D and back again.

1.9 Explain Example 1.3 in terms of maps.

✓ 1.10 [Halmos] Are there two matrices A and B that are similar while A^2 and B^2 are not similar?

✓ 1.11 Prove that if two matrices are similar and one is invertible then so is the other.

✓ 1.12 Show that similarity is an equivalence relation. (The definition given earlier already reflects this, so instead start here with the definition that $\hat{T}$ is similar to T if $\hat{T} = PTP^{-1}$.)

1.13 Consider a matrix representing, with respect to some B, B, reflection across the x-axis in $\mathbb{R}^2$. Consider also a matrix representing, with respect to some D, D, reflection across the y-axis. Must they be similar?

1.14 Prove that similarity preserves determinants and rank. Does the converse hold?

1.15 Is there a matrix equivalence class with only one matrix similarity class inside? One with infinitely many similarity classes?

1.16 Can two different diagonal matrices be in the same similarity class?

✓ 1.17 Prove that if two matrices are similar then their k-th powers are similar when $k > 0$. What if $k \leqslant 0$?

✓ 1.18 Let $p(x)$ be the polynomial $c_n x^n + \cdots + c_1 x + c_0$. Show that if T is similar to S then $p(T) = c_n T^n + \cdots + c_1 T + c_0 I$ is similar to $p(S) = c_n S^n + \cdots + c_1 S + c_0 I$.

1.19 List all of the matrix equivalence classes of $1{\times}1$ matrices. Also list the similarity classes, and describe which similarity classes are contained inside of each matrix equivalence class.

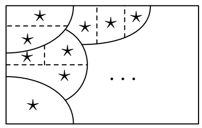
1.20 Does similarity preserve sums?

1.21 Show that if $T - \lambda I$ and N are similar matrices then T and $N + \lambda I$ are also similar.

II.2 Diagonalizability

The prior subsection shows that although similar matrices are necessarily matrix equivalent, the converse does not hold. Some matrix equivalence classes break into two or more similarity classes; for instance, the nonsingular 2×2 matrices form one matrix equivalence class but more than one similarity class.

Thus we cannot use the canonical form for matrix equivalence, a block partial-identity matrix, as a canonical form for matrix similarity. The diagram below illustrates. The stars are similarity class representatives. Each dashed-line similarity class subdivision has one star but each solid-curve matrix equivalence class division has only one partial identity matrix.



To develop a canonical form for representatives of the similarity classes we naturally build on previous work. This means first that the partial identity matrices should represent the similarity classes into which they fall. Beyond that, the representatives should be as simple as possible. The simplest extension of the partial identity form is the diagonal form.

2.1 Definition A transformation is *diagonalizable* if it has a diagonal representation with respect to the same basis for the codomain as for the domain. A *diagonalizable matrix* is one that is similar to a diagonal matrix: T is diagonalizable if there is a nonsingular P such that PTP^{-1} is diagonal.

2.2 Example The matrix

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

is diagonalizable.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

2.3 Example We will show that this matrix is not diagonalizable.

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The fact that N is not the zero matrix means that it cannot be similar to the zero matrix, because the zero matrix is similar only to itself. Thus if N were to be similar to a diagonal matrix then that matrix would have have at least one nonzero entry on its diagonal.

The square of N is the zero matrix. This imples that for any map n represented by N (with respect to some B, B) the composition $n \circ n$ is the zero map. This in turn implies that for any matrix representing n (with respect to some $\hat{B}, \hat{B}$), its square is the zero matrix. But the square of a nonzero diagonal matrix cannot be the zero matrix, because the square of a diagonal matrix is the diagonal matrix whose entries are the squares of the entries from the starting matrix. Thus there is no $\hat{B}, \hat{B}$ such that n is represented by a diagonal matrix — the matrix N is not diagonalizable.

That example shows that a diagonal form will not suffice as a canonical form for similarity — we cannot find a diagonal matrix in each matrix similarity class. However, some similarity classes contain a diagonal matrix and the canonical form that we are developing has the property that if a matrix can be diagonalized then the diagonal matrix is the canonical representative of its similarity class.

2.4 Lemma A transformation t is diagonalizable if and only if there is a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and scalars $\lambda_1, \ldots, \lambda_n$ such that $t(\vec{\beta}_i) = \lambda_i \vec{\beta}_i$ for each i.

Proof Consider a diagonal representation matrix.

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} \vdots & & \vdots \\ \mathrm{Rep}_B(t(\vec{\beta}_1)) & \cdots & \mathrm{Rep}_B(t(\vec{\beta}_n)) \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}$$

Consider the representation of a member of this basis with respect to the basis $\mathrm{Rep}_B(\vec{\beta}_i)$. The product of the diagonal matrix and the representation vector

$$\mathrm{Rep}_B(t(\vec{\beta}_i)) = \begin{pmatrix} \lambda_1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix}$$

has the stated action. QED

2.5 Example To diagonalize

$$T = \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}$$

we take T as the representation of a transformation with respect to the standard basis $\mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ and look for a basis $B=\langle\vec{\beta}_1,\vec{\beta}_2\rangle$ such that

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$$

that is, such that $t(\vec{\beta}_1)=\lambda_1\vec{\beta}_1$ and $t(\vec{\beta}_2)=\lambda_2\vec{\beta}_2$.

$$\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}\vec{\beta}_1=\lambda_1\cdot\vec{\beta}_1 \qquad \begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}\vec{\beta}_2=\lambda_2\cdot\vec{\beta}_2$$

We are looking for scalars x such that this equation

$$\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}=x\cdot\begin{pmatrix}b_1\\ b_2\end{pmatrix}$$

has solutions b_1 and b_2 that are not both 0 (the zero vector is not the member of any basis). That's a linear system.

$$\begin{aligned}(3-x)\cdot b_1+\phantom{(1-x)\cdot{}}2\cdot b_2&=0\\ (1-x)\cdot b_2&=0\end{aligned}\qquad(*)$$

Focus first on the bottom equation. There are two cases: either $b_2=0$ or $x=1$.

In the $b_2=0$ case the first equation gives that either $b_1=0$ or $x=3$. Since we've disallowed the possibility that both $b_2=0$ and $b_1=0$, we are left with the first diagonal entry $\lambda_1=3$. With that, $(*)$'s first equation is $0\cdot b_1+2\cdot b_2=0$ and so associated with $\lambda_1=3$ are vectors having a second component of zero while the first component is free.

$$\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}b_1\\ 0\end{pmatrix}=3\cdot\begin{pmatrix}b_1\\ 0\end{pmatrix}$$

To get a first basis vector choose any nonzero b_1.

$$\vec{\beta}_1=\begin{pmatrix}1\\ 0\end{pmatrix}$$

The other case for the bottom equation of $(*)$ is $\lambda_2=1$. Then $(*)$'s first equation is $2\cdot b_1+2\cdot b_2=0$ and so associated with this case are vectors whose second component is the negative of the first.

$$\begin{pmatrix}3 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}b_1\\ -b_1\end{pmatrix}=1\cdot\begin{pmatrix}b_1\\ -b_1\end{pmatrix}$$

Get the second basis vector by choosing a nonzero one of these.

$$\vec{\beta}_2=\begin{pmatrix}1\\ -1\end{pmatrix}$$

Now draw the similarity diagram

$$\begin{array}{ccc} \mathbb{R}^2_{wrt\ \mathcal{E}_2} & \xrightarrow[T]{t} & \mathbb{R}^2_{wrt\ \mathcal{E}_2} \\ {\scriptstyle\text{id}}\big\downarrow & & {\scriptstyle\text{id}}\big\downarrow \\ \mathbb{R}^2_{wrt\ B} & \xrightarrow[D]{t} & \mathbb{R}^2_{wrt\ B} \end{array}$$

and note that the matrix $\text{Rep}_{B,\mathcal{E}_2}(\text{id})$ is easy, giving this diagonalization.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

In the next subsection we will expand on that example by considering more closely the property of Lemma 2.4. This includes seeing a streamlined way to find the λ's.

Exercises

✓ **2.6** Repeat Example 2.5 for the matrix from Example 2.2.

2.7 Diagonalize these upper triangular matrices.

(a) $\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$

✓ **2.8** What form do the powers of a diagonal matrix have?

2.9 Give two same-sized diagonal matrices that are not similar. Must any two different diagonal matrices come from different similarity classes?

2.10 Give a nonsingular diagonal matrix. Can a diagonal matrix ever be singular?

✓ **2.11** Show that the inverse of a diagonal matrix is the diagonal of the the inverses, if no element on that diagonal is zero. What happens when a diagonal entry is zero?

2.12 The equation ending Example 2.5

$$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

is a bit jarring because for P we must take the first matrix, which is shown as an inverse, and for P^{-1} we take the inverse of the first matrix, so that the two -1 powers cancel and this matrix is shown without a superscript -1.

(a) Check that this nicer-appearing equation holds.

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}^{-1}$$

(b) Is the previous item a coincidence? Or can we always switch the P and the P^{-1}?

2.13 Show that the P used to diagonalize in Example 2.5 is not unique.

2.14 Find a formula for the powers of this matrix *Hint*: see Exercise 8.

$$\begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix}$$

✓ **2.15** Diagonalize these.

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ **(b)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

2.16 We can ask how diagonalization interacts with the matrix operations. Assume that $t, s\colon V \to V$ are each diagonalizable. Is ct diagonalizable for all scalars c? What about $t + s$? $t \circ s$?

✓ 2.17 Show that matrices of this form are not diagonalizable.

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad c \neq 0$$

2.18 Show that each of these is diagonalizable.

(a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **(b)** $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ $\quad x, y, z$ scalars

II.3 Eigenvalues and Eigenvectors

We will next focus on the property of Lemma 2.4.

3.1 Definition A transformation $t\colon V \to V$ has a scalar *eigenvalue* λ if there is a nonzero *eigenvector* $\vec{\zeta} \in V$ such that $t(\vec{\zeta}) = \lambda \cdot \vec{\zeta}$.

("Eigen" is German for "characteristic of" or "peculiar to." Some authors call these *characteristic* values and vectors. No authors call them "peculiar.")

3.2 Example The projection map

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \qquad x, y, z \in \mathbb{C}$$

has an eigenvalue of 1 associated with any eigenvector

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

where x and y are scalars that are not both zero.

In contrast, a number that is not an eigenvalue of of this map is 2, since assuming that π doubles a vector leads to the equations $x = 2x$, $y = 2y$, and $0 = 2z$, and thus no non-$\vec{0}$ vector is doubled.

Note that the definition requires that the eigenvector be non-$\vec{0}$. Some authors allow $\vec{0}$ as an eigenvector for λ as long as there are also non-$\vec{0}$ vectors associated with λ. The key point is to disallow the trivial case where λ is such that $t(\vec{v}) = \lambda\vec{v}$ for only the single vector $\vec{v} = \vec{0}$.

Also, note that the eigenvalue λ could be 0. The issue is whether $\vec{\zeta}$ equals $\vec{0}$.

3.3 Example The only transformation on the trivial space $\{\vec{0}\}$ is $\vec{0} \mapsto \vec{0}$. This map has no eigenvalues because there are no non-$\vec{0}$ vectors $\vec{v}$ mapped to a scalar multiple $\lambda \cdot \vec{v}$ of themselves.

3.4 Example Consider the homomorphism $t\colon \mathcal{P}_1 \to \mathcal{P}_1$ given by $c_0 + c_1x \mapsto (c_0 + c_1) + (c_0 + c_1)x$. While the codomain $\mathcal{P}_1$ of t is two-dimensional, its range is one-dimensional $\mathscr{R}(t) = \{c + cx \mid c \in \mathbb{C}\}$. Application of t to a vector in that range will simply rescale the vector $c + cx \mapsto (2c) + (2c)x$. That is, t has an eigenvalue of 2 associated with eigenvectors of the form $c + cx$ where $c \neq 0$.

This map also has an eigenvalue of 0 associated with eigenvectors of the form $c - cx$ where $c \neq 0$.

The definition above is for maps. We can give a matrix version.

3.5 Definition A square matrix T has a scalar *eigenvalue* λ associated with the nonzero *eigenvector* $\vec{\zeta}$ if $T\vec{\zeta} = \lambda \cdot \vec{\zeta}$.

This extension of the definition for maps to a definition for matrices is natural but there is a point on which we must take care. The eigenvalues of a map are also the eigenvalues of matrices r epresenting that map, and so similar matrices have the same eigenvalues. However, the eigenvectors can differ — similar matrices need not have the same eigenvectors. The next example explains.

3.6 Example These matrices are similar

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{T} = \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$$

since $\hat{T} = PTP^{-1}$ for this P.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

The matrix T has two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 0$. The first one is associated with this eigenvector.

$$T\vec{e}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\vec{e}_1$$

Suppose that T represents a transformation $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis. Then the action of this transformation t is simple.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} 2x \\ 0 \end{pmatrix}$$

Of course, $\hat{T}$ represents the same transformation but with respect to a different basis B. We can easily find this basis. The arrow diagram

$$\begin{array}{ccc} V_{\mathit{wrt}\ \mathcal{E}_3} & \xrightarrow[T]{t} & V_{\mathit{wrt}\ \mathcal{E}_3} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ V_{\mathit{wrt}\ B} & \xrightarrow[\hat{T}]{t} & V_{\mathit{wrt}\ B} \end{array}$$

shows that $P^{-1} = \mathrm{Rep}_{B,\mathcal{E}_3}(\mathrm{id})$. By the definition of the matrix representation of a map, its first column is $\mathrm{Rep}_{\mathcal{E}_3}(\mathrm{id}(\vec{\beta}_1)) = \mathrm{Rep}_{\mathcal{E}_3}(\vec{\beta}_1)$. With respect to the standard basis any vector is represented by itself, so the first basis element $\vec{\beta}_1$ is the first column of P^{-1}. The same goes for the other one.

$$B = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rangle$$

Since the matrices T and $\hat{T}$ both represent the transformation t, both reflect the action $t(\vec{e}_1) = 2\vec{e}_1$.

$$\begin{aligned} \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t) \cdot \mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) &= T \cdot \mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) = 2 \cdot \mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) \\ \mathrm{Rep}_{B,B}(t) \cdot \mathrm{Rep}_{B}(\vec{e}_1) &= \hat{T} \cdot \mathrm{Rep}_{B}(\vec{e}_1) = 2 \cdot \mathrm{Rep}_{B}(\vec{e}_1) \end{aligned}$$

But while in those two equations the eigenvalue 2's are the same, the vector representations differ.

$$\begin{aligned} T \cdot \mathrm{Rep}_{\mathcal{E}_2}(\vec{e}_1) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \hat{T} \cdot \mathrm{Rep}_{B}(\vec{e}_1) = \hat{T} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

That is, when the matrix representing the transformation is $T = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(t)$ then it "assumes" that column vectors are representations with respect to $\mathcal{E}_2$. However $\hat{T} = \mathrm{Rep}_{B,B}(t)$ "assumes" that column vectors are representations with respect to B, and so the column vectors that get doubled are different.

We next see the basic tool for finding eigenvectors and eigenvalues.

3.7 Example If

$$T = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}$$

then to find the scalars x such that $T\vec{\zeta}=x\vec{\zeta}$ for nonzero eigenvectors $\vec{\zeta}$, bring everything to the left-hand side

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} - x\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \vec{0}$$

and factor $(T-xI)\vec{\zeta}=\vec{0}$. (Note that it says $T-xI$. The expression $T-x$ doesn't make sense because T is a matrix while x is a scalar.) This homogeneous linear system

$$\begin{pmatrix} 1-x & 2 & 1 \\ 2 & 0-x & -2 \\ -1 & 2 & 3-x \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has a nonzero solution $\vec{z}$ if and only if the matrix is singular. We can determine when that happens.

$$\begin{aligned} 0 &= |T-xI| \\ &= \begin{vmatrix} 1-x & 2 & 1 \\ 2 & 0-x & -2 \\ -1 & 2 & 3-x \end{vmatrix} \\ &= x^3-4x^2+4x \\ &= x(x-2)^2 \end{aligned}$$

The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$. To find the associated eigenvectors plug in each eigenvalue. Plugging in $\lambda_1=0$ gives

$$\begin{pmatrix} 1-0 & 2 & 1 \\ 2 & 0-0 & -2 \\ -1 & 2 & 3-0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a \\ -a \\ a \end{pmatrix}$$

for $a \neq 0$ (a must be non-0 because eigenvectors are defined to be non-$\vec{0}$). Plugging in $\lambda_2=2$ gives

$$\begin{pmatrix} 1-2 & 2 & 1 \\ 2 & 0-2 & -2 \\ -1 & 2 & 3-2 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} b \\ 0 \\ b \end{pmatrix}$$

with $b \neq 0$.

3.8 Example If

$$S=\begin{pmatrix} \pi & 1 \\ 0 & 3 \end{pmatrix}$$

(here π is not a projection map, it is the number $3.14\ldots$) then

$$\begin{vmatrix} \pi - x & 1 \\ 0 & 3 - x \end{vmatrix} = (x - \pi)(x - 3)$$

so S has eigenvalues of $\lambda_1 = \pi$ and $\lambda_2 = 3$. To find associated eigenvectors, first plug in λ_1 for x

$$\begin{pmatrix} \pi - \pi & 1 \\ 0 & 3 - \pi \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

for a scalar $a \neq 0$. Then plug in λ_2

$$\begin{pmatrix} \pi - 3 & 1 \\ 0 & 3 - 3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -b/(\pi - 3) \\ b \end{pmatrix}$$

where $b \neq 0$.

3.9 Definition The *characteristic polynomial of a square matrix* T is the determinant $|T - xI|$ where x is a variable. The *characteristic equation* is $|T - xI| = 0$. The *characteristic polynomial of a transformation* t is the characteristic polynomial of any matrix representation $\mathrm{Rep}_{B,B}(t)$.

Exercise 30 checks that the characteristic polynomial of a transformation is well-defined, that is, that the characteristic polynomial is the same no matter which basis we use for the representation.

3.10 Lemma A linear transformation on a nontrivial vector space has at least one eigenvalue.

PROOF Any root of the characteristic polynomial is an eigenvalue. Over the complex numbers, any polynomial of degree one or greater has a root. QED

3.11 Remark That result is the reason that in this chapter we use scalars that are complex numbers.

3.12 Definition The *eigenspace of a transformation* t *associated with the eigenvalue* λ is $V_\lambda = \{\vec{\zeta} \mid t(\vec{\zeta}\,) = \lambda\vec{\zeta}\,\}$. The eigenspace of a matrix is analogous.

3.13 Lemma An eigenspace is a subspace.

Proof Fix an eigenvalue λ. Notice first that V_λ contains the zero vector since $t(\vec{0})=\vec{0}$, which equals $\lambda\vec{0}$. So the eigenspace is a nonempty subset of the space. What remains is to check closure of this set under linear combinations. Take $\vec{\zeta}_1,\ldots,\vec{\zeta}_n\in V_\lambda$ and then verify

$$\begin{aligned} t(c_1\vec{\zeta}_1+c_2\vec{\zeta}_2+\cdots+c_n\vec{\zeta}_n) &= c_1t(\vec{\zeta}_1)+\cdots+c_nt(\vec{\zeta}_n)\\ &= c_1\lambda\vec{\zeta}_1+\cdots+c_n\lambda\vec{\zeta}_n\\ &= \lambda(c_1\vec{\zeta}_1+\cdots+c_n\vec{\zeta}_n)\end{aligned}$$

that the combination is also an element of V_λ. QED

3.14 Example In Example 3.7 these are the eigenspaces associated with the eigenvalues 0 and 2.

$$V_0=\{\begin{pmatrix}a\\-a\\a\end{pmatrix}\mid a\in\mathbb{C}\},\qquad V_2=\{\begin{pmatrix}b\\0\\b\end{pmatrix}\mid b\in\mathbb{C}\}.$$

3.15 Example In Example 3.8 these are the eigenspaces associated with the eigenvalues π and 3.

$$V_\pi=\{\begin{pmatrix}a\\0\end{pmatrix}\mid a\in\mathbb{C}\}\qquad V_3=\{\begin{pmatrix}-b/(\pi-3)\\b\end{pmatrix}\mid b\in\mathbb{C}\}$$

The characteristic equation in Example 3.7 is $0=x(x-2)^2$ so in some sense 2 is an eigenvalue twice. However there are not twice as many eigenvectors in that the dimension of the associated eigenspace V_2 is one, not two. The next example is a case where a number is a double root of the characteristic equation and the dimension of the associated eigenspace is two.

3.16 Example With respect to the standard bases, this matrix

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

represents projection.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\stackrel{\pi}{\longmapsto}\begin{pmatrix}x\\y\\0\end{pmatrix}\qquad x,y,z\in\mathbb{C}$$

Its characteristic equation

$$\begin{aligned} 0 &= |T - xI| \\ &= \begin{vmatrix} 1-x & 0 & 0 \\ 0 & 1-x & 0 \\ 0 & 0 & 0-x \end{vmatrix} \\ &= (1-x)^2(0-x) \end{aligned}$$

has the double root $x = 1$ along with the single root $x = 0$. Its eigenspace associated with the eigenvalue 0 and its eigenspace associated with the eigenvalue 1 are easy to find.

$$V_0 = \{\begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \mid c_3 \in \mathbb{C}\} \qquad V_1 = \{\begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \mid c_1, c_2 \in \mathbb{C}\}$$

Note that V_1 has dimension two.

By Lemma 3.13 if two eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are associated with the same eigenvalue then a linear combination of those two is also an eigenvector, associated with the same eigenvalue. As an illustration, referring to the prior example, this sum of two members of V_1

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

yields another member of V_1.

The next result speaks to the situation where the vectors come from different eigenspaces.

3.17 Theorem For any set of distinct eigenvalues of a map or matrix, a set of associated eigenvectors, one per eigenvalue, is linearly independent.

Proof We will use induction on the number of eigenvalues. The base step is that there are zero eigenvalues. Then the set of associated vectors is empty and so is linearly independent.

For the inductive step assume that the statement is true for any set of $k \geqslant 0$ distinct eigenvalues. Consider distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$ and let $\vec{v}_1, \ldots, \vec{v}_{k+1}$ be associated eigenvectors. Suppose that $\vec{0} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1}$. Derive two equations from that, the first by multiplying by λ_{k+1} on both sides $\vec{0} = c_1\lambda_{k+1}\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ and the second by applying the map to both sides $\vec{0} = c_1 t(\vec{v}_1) + \cdots + c_{k+1} t(\vec{v}_{k+1}) = c_1\lambda_1\vec{v}_1 + \cdots + c_{k+1}\lambda_{k+1}\vec{v}_{k+1}$ (applying the matrix gives the same result). Subtract the second from the first.

$$\vec{0} = c_1(\lambda_{k+1} - \lambda_1)\vec{v}_1 + \cdots + c_k(\lambda_{k+1} - \lambda_k)\vec{v}_k + c_{k+1}(\lambda_{k+1} - \lambda_{k+1})\vec{v}_{k+1}$$

The $\vec{v}_{k+1}$ term vanishes. Then the induction hypothesis gives that $c_1(\lambda_{k+1} - \lambda_1) = 0, \ldots, c_k(\lambda_{k+1} - \lambda_k) = 0$. The eigenvalues are distinct so the coefficients $c_1, \ldots, c_k$ are all 0. With that we are left with the equation $\vec{0} = c_{k+1}\vec{v}_{k+1}$ so c_{k+1} is also 0. QED

3.18 Example The eigenvalues of

$$\begin{pmatrix} 2 & -2 & 2 \\ 0 & 1 & 1 \\ -4 & 8 & 3 \end{pmatrix}$$

are distinct: $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. A set of associated eigenvectors

$$\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\}$$

is linearly independent.

3.19 Corollary An $n\times n$ matrix with n distinct eigenvalues is diagonalizable.

Proof Form a basis of eigenvectors. Apply Lemma 2.4. QED

This section observes that some matrices are similar to a diagonal matrix. The idea of eigenvalues arose as the entries of that diagonal matrix, although the definition applies more broadly than just to diagonalizable matrices. To find eigenvalues we defined the characteristic equation and that led to the final result, a criteria for diagonalizability. (While it is useful for the theory, note that in applications finding eigenvalues this way is typically impractical; for one thing the matrix may be large and finding roots of large-degree polynomials is hard.)

In the next section we study matrices that cannot be diagonalized.

Exercises

3.20 For each, find the characteristic polynomial and the eigenvalues.

(a) $\begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 3 \\ 7 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
(e) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

✓ 3.21 For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

(a) $\begin{pmatrix}3&0\\8&-1\end{pmatrix}$ **(b)** $\begin{pmatrix}3&2\\-1&0\end{pmatrix}$

3.22 Find the characteristic equation, and the eigenvalues and associated eigenvectors for this matrix. *Hint.* The eigenvalues are complex.

$$\begin{pmatrix}-2&-1\\5&2\end{pmatrix}$$

3.23 Find the characteristic polynomial, the eigenvalues, and the associated eigenvectors of this matrix.

$$\begin{pmatrix}1&1&1\\0&0&1\\0&0&1\end{pmatrix}$$

✓ 3.24 For each matrix, find the characteristic equation, and the eigenvalues and associated eigenvectors.

(a) $\begin{pmatrix}3&-2&0\\-2&3&0\\0&0&5\end{pmatrix}$ **(b)** $\begin{pmatrix}0&1&0\\0&0&1\\4&-17&8\end{pmatrix}$

✓ 3.25 Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be

$$a_0 + a_1x + a_2x^2 \mapsto (5a_0 + 6a_1 + 2a_2) - (a_1 + 8a_2)x + (a_0 - 2a_2)x^2.$$

Find its eigenvalues and the associated eigenvectors.

3.26 Find the eigenvalues and eigenvectors of this map $t\colon \mathcal{M}_2 \to \mathcal{M}_2$.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix} \mapsto \begin{pmatrix}2c&a+c\\b-2c&d\end{pmatrix}$$

✓ 3.27 Find the eigenvalues and associated eigenvectors of the differentiation operator $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$.

3.28 Prove that the eigenvalues of a triangular matrix (upper or lower triangular) are the entries on the diagonal.

✓ 3.29 Find the formula for the characteristic polynomial of a 2×2 matrix.

3.30 Prove that the characteristic polynomial of a transformation is well-defined.

3.31 Prove or disprove: if all the eigenvalues of a matrix are 0 then it must be the zero matrix.

✓ 3.32 **(a)** Show that any non-$\vec{0}$ vector in any nontrivial vector space can be a eigenvector. That is, given a $\vec{v} \neq \vec{0}$ from a nontrivial V, show that there is a transformation $t\colon V \to V$ having a scalar eigenvalue $\lambda \in \mathbb{R}$ such that $\vec{v} \in V_\lambda$.

(b) What if we are given a scalar λ? Can any non-$\vec{0}$ member of any nontrivial vector space be an eigenvector associated with λ?

✓ 3.33 Suppose that $t\colon V \to V$ and $T = \text{Rep}_{B,B}(t)$. Prove that the eigenvectors of T associated with λ are the non-$\vec{0}$ vectors in the kernel of the map represented (with respect to the same bases) by $T - \lambda I$.

3.34 Prove that if $a, \ldots, d$ are all integers and $a + b = c + d$ then

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

has integral eigenvalues, namely $a + b$ and $a - c$.

✓ 3.35 Prove that if T is nonsingular and has eigenvalues $\lambda_1, \ldots, \lambda_n$ then T^{-1} has eigenvalues $1/\lambda_1, \ldots, 1/\lambda_n$. Is the converse true?

✓ 3.36 Suppose that T is $n \times n$ and c, d are scalars.

(a) Prove that if T has the eigenvalue λ with an associated eigenvector $\vec{v}$ then $\vec{v}$ is an eigenvector of $cT + dI$ associated with eigenvalue $c\lambda + d$.

(b) Prove that if T is diagonalizable then so is $cT + dI$.

✓ 3.37 Show that λ is an eigenvalue of T if and only if the map represented by $T - \lambda I$ is not an isomorphism.

3.38 [Strang 80]

(a) Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k.

(b) What is wrong with this proof generalizing that? "If λ is an eigenvalue of A and μ is an eigenvalue for B, then $\lambda\mu$ is an eigenvalue for AB, for, if $A\vec{x} = \lambda\vec{x}$ and $B\vec{x} = \mu\vec{x}$ then $AB\vec{x} = A\mu\vec{x} = \mu A\vec{x} = \mu\lambda\vec{x}$"?

3.39 Do matrix equivalent matrices have the same eigenvalues?

3.40 Show that a square matrix with real entries and an odd number of rows has at least one real eigenvalue.

3.41 Diagonalize.

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$$

3.42 Suppose that P is a nonsingular $n \times n$ matrix. Show that the *similarity transformation* map $t_P \colon \mathcal{M}_{n\times n} \to \mathcal{M}_{n\times n}$ sending $T \mapsto PTP^{-1}$ is an isomorphism.

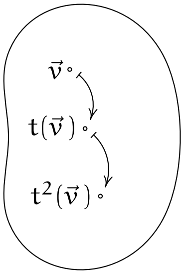
? 3.43 [Math. Mag., Nov. 1967] Show that if A is an n square matrix and each row (column) sums to c then c is a characteristic root of A. ("Characteristic root" is a synonym for eigenvalue.)

III Nilpotence

This chapter shows that every square matrix is similar to one that is a sum of two kinds of simple matrices. The prior section focused on the first simple kind, diagonal matrices. We now consider the other kind.

III.1 Self-Composition

Because a linear transformation $t\colon V \to V$ has the same domain as codomain, we can compose t with itself $t^2 = t \circ t$, and $t^3 = t \circ t \circ t$, etc.*



Note that the superscript power notation t^j for iterates of the transformations fits with the notation that we've used for their square matrix representations because if $\mathrm{Rep}_{B,B}(t) = T$ then $\mathrm{Rep}_{B,B}(t^j) = T^j$.

1.1 Example For the derivative map $d/dx\colon \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$a + bx + cx^2 + dx^3 \stackrel{d/dx}{\longmapsto} b + 2cx + 3dx^2$$

the second power is the second derivative

$$a + bx + cx^2 + dx^3 \stackrel{d^2/dx^2}{\longmapsto} 2c + 6dx$$

the third power is the third derivative

$$a + bx + cx^2 + dx^3 \stackrel{d^3/dx^3}{\longmapsto} 6d$$

and any higher power is the zero map.

1.2 Example This transformation of the space $\mathcal{M}_{2\times 2}$ of 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{t}{\longmapsto} \begin{pmatrix} b & a \\ d & 0 \end{pmatrix}$$

* More information on function iteration is in the appendix.

has this second power

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{t^2} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

and this third power.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{t^3} \begin{pmatrix} b & a \\ 0 & 0 \end{pmatrix}$$

After that, $t^4 = t^2$ and $t^5 = t^3$, etc.

1.3 Example Consider the shift transformation $t\colon \mathbb{C}^3 \to \mathbb{C}^3$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{t} \begin{pmatrix} 0 \\ x \\ y \end{pmatrix}$$

We have that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \xmapsto{t} \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} \xmapsto{t} \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix} \xmapsto{t} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so the range spaces descend to the trivial subspace.

$$\mathscr{R}(t) = \{ \begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \mid a, b \in \mathbb{C} \} \qquad \mathscr{R}(t^2) = \{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \mid c \in \mathbb{C} \} \qquad \mathscr{R}(t^3) = \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \}$$

These examples suggest that after some number of iterations the map settles down.

1.4 Lemma For any transformation $t\colon V \to V$, the range spaces of the powers form a descending chain

$$V \supseteq \mathscr{R}(t) \supseteq \mathscr{R}(t^2) \supseteq \cdots$$

and the null spaces form an ascending chain.

$$\{\vec{0}\} \subseteq \mathscr{N}(t) \subseteq \mathscr{N}(t^2) \subseteq \cdots$$

Further, there is a k such that for powers less than k the subsets are proper so that if $j < k$ then $\mathscr{R}(t^j) \supset \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) \subset \mathscr{N}(t^{j+1})$, while for higher powers the sets are equal so that if $j \geqslant k$ then $\mathscr{R}(t^j) = \mathscr{R}(t^{j+1})$ and $\mathscr{N}(t^j) = \mathscr{N}(t^{j+1})$).

Proof First recall that for any map the dimension of its range space plus the dimension of its null space equals the dimension of its domain. So if the dimensions of the range spaces shrink then the dimensions of the null spaces must rise. We will do the range space half here and leave the rest for Exercise 14.

We start by showing that the range spaces form a chain. If $\vec{w} \in \mathscr{R}(t^{j+1})$, so that $\vec{w} = t^{j+1}(\vec{v})$ for some $\vec{v}$, then $\vec{w} = t^{j}(\,t(\vec{v})\,)$. Thus $\vec{w} \in \mathscr{R}(t^{j})$.

Next we verify the "further" property: in the chain the subsets containments are proper initially, and then from some power k onward the range spaces are equal. We first show that if any pair of adjacent range spaces in the chain are equal $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1})$ then all subsequent ones are also equal $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$, etc. This holds because $t\colon \mathscr{R}(t^{k+1}) \to \mathscr{R}(t^{k+2})$ is the same map, with the same domain, as $t\colon \mathscr{R}(t^{k}) \to \mathscr{R}(t^{k+1})$ and it therefore has the same range $\mathscr{R}(t^{k+1}) = \mathscr{R}(t^{k+2})$ (it holds for all higher powers by induction). So if the chain of range spaces ever stops strictly decreasing then from that point onward it is stable.

We end by showing that the chain must eventually stop decreasing. Each range space is a subspace of the one before it. For it to be a proper subspace it must be of strictly lower dimension (see Exercise 12). These spaces are finite-dimensional and so the chain can fall for only finitely many steps. That is, the power k is at most the dimension of V. QED

1.5 Example The derivative map $a + bx + cx^2 + dx^3 \xrightarrow{d/dx} b + 2cx + 3dx^2$ on $\mathcal{P}_3$ has this chain of range spaces

$$\mathscr{R}(t^0) = \mathcal{P}_3 \supset \mathscr{R}(t^1) = \mathcal{P}_2 \supset \mathscr{R}(t^2) = \mathcal{P}_1 \supset \mathscr{R}(t^3) = \mathcal{P}_0 \supset \mathscr{R}(t^4) = \{\vec{0}\}$$

(all later elements of the chain are the trivial space). And it has this chain of null spaces

$$\mathscr{N}(t^0) = \{\vec{0}\} \subset \mathscr{N}(t^1) = \mathcal{P}_0 \subset \mathscr{N}(t^2) = \mathcal{P}_1 \subset \mathscr{N}(t^3) = \mathcal{P}_2 \subset \mathscr{N}(t^4) = \mathcal{P}_3$$

(later elements are the entire space).

1.6 Example Let $t\colon \mathcal{P}_2 \to \mathcal{P}_2$ be the map $c_0 + c_1x + c_2x^2 \mapsto 2c_0 + c_2x$. As the lemma describes, on iteration the range space shrinks

$$\mathscr{R}(t^0) = \mathcal{P}_2 \quad \mathscr{R}(t) = \{\,a + bx \mid a, b \in \mathbb{C}\,\} \quad \mathscr{R}(t^2) = \{\,a \mid a \in \mathbb{C}\,\}$$

and then stabilizes $\mathscr{R}(t^2) = \mathscr{R}(t^3) = \cdots$ while the null space grows

$$\mathscr{N}(t^0) = \{0\} \quad \mathscr{N}(t) = \{\,cx \mid c \in \mathbb{C}\,\} \quad \mathscr{N}(t^2) = \{\,cx + d \mid c, d \in \mathbb{C}\,\}$$

and then stabilizes $\mathscr{N}(t^2) = \mathscr{N}(t^3) = \cdots$.

1.7 Example The transformation $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates

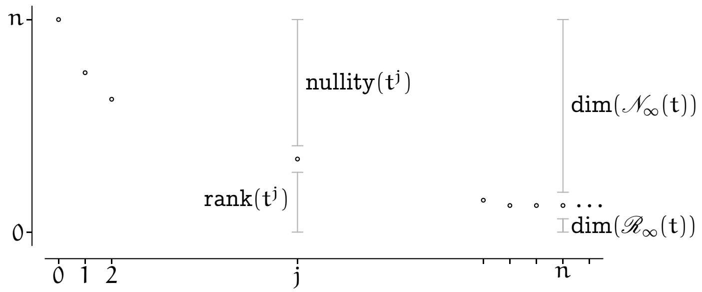
$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$$

has $\mathbb{C}^3 \supset \mathscr{R}(\pi) = \mathscr{R}(\pi^2) = \cdots$ and $\{\vec{0}\} \subset \mathscr{N}(\pi) = \mathscr{N}(\pi^2) = \cdots$ where this is the range space and the null space.

$$\mathscr{R}(\pi) = \{\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \mid a, b \in \mathbb{C}\} \qquad \mathscr{N}(\pi) = \{\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \mid c \in \mathbb{C}\}$$

1.8 Definition Let t be a transformation on an n-dimensional space. The *generalized range space* (or *closure of the range space*) is $\mathscr{R}_\infty(t) = \mathscr{R}(t^n)$. The *generalized null space* (or *closure of the null space*) is $\mathscr{N}_\infty(t) = \mathscr{N}(t^n)$.

This graph illustrates. The horizontal axis gives the power j of a transformation. The vertical axis gives the dimension of the range space of t^j as the distance above zero, and thus also shows the dimension of the null space because the two add to the dimension n of the domain.



On iteration the rank falls and the nullity rises until there is some k such that the map reaches a steady state $\mathscr{R}(t^k) = \mathscr{R}(t^{k+1}) = \mathscr{R}_\infty(t)$ and $\mathscr{N}(t^k) = \mathscr{N}(t^{k+1}) = \mathscr{N}_\infty(t)$. This must happen by the n-th iterate.

Exercises

✓ 1.9 Give the chains of range spaces and null spaces for the zero and identity transformations.

✓ 1.10 For each map, give the chain of range spaces and the chain of null spaces, and the generalized range space and the generalized null space.

(a) $t_0 \colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx^2$

(b) $t_1 \colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ a \end{pmatrix}$$

(c) $t_2 \colon \mathcal{P}_2 \to \mathcal{P}_2$, $a + bx + cx^2 \mapsto b + cx + ax^2$

(d) $t_3 \colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a \\ a \\ b \end{pmatrix}$$

1.11 Prove that function composition is associative $(t \circ t) \circ t = t \circ (t \circ t)$ and so we can write t^3 without specifying a grouping.

1.12 Check that a subspace must be of dimension less than or equal to the dimension of its superspace. Check that if the subspace is proper (the subspace does not equal the superspace) then the dimension is strictly less. *(This is used in the proof of Lemma 1.4.)*

✓ 1.13 Prove that the generalized range space $\mathscr{R}_\infty(t)$ is the entire space, and the generalized null space $\mathscr{N}_\infty(t)$ is trivial, if the transformation t is nonsingular. Is this 'only if' also?

1.14 Verify the null space half of Lemma 1.4.

✓ 1.15 Give an example of a transformation on a three dimensional space whose range has dimension two. What is its null space? Iterate your example until the range space and null space stabilize.

1.16 Show that the range space and null space of a linear transformation need not be disjoint. Are they ever disjoint?

III.2 Strings

This requires material from the optional Combining Subspaces subsection.

The prior subsection shows that as j increases the dimensions of the $\mathscr{R}(t^j)$'s fall while the dimensions of the $\mathscr{N}(t^j)$'s rise, in such a way that this rank and nullity split between them the dimension of V. Can we say more; do the two split a basis — is $V = \mathscr{R}(t^j) \oplus \mathscr{N}(t^j)$?

The answer is yes for the smallest power $j = 0$ since $V = \mathscr{R}(t^0) \oplus \mathscr{N}(t^0) = V \oplus \{\vec{0}\}$. The answer is also yes at the other extreme.

2.1 Lemma For any linear $t\colon V \to V$ the function $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one.

Proof Let the dimension of V be n. Because $\mathscr{R}(t^n) = \mathscr{R}(t^{n+1})$, the map $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is a dimension-preserving homomorphism. Therefore, by Theorem Two.II.2.20 it is one-to-one. QED

2.2 Corollary Where $t\colon V \to V$ is a linear transformation, the space is the direct sum $V = \mathscr{R}_\infty(t) \oplus \mathscr{N}_\infty(t)$. That is, both (1) $\dim(V) = \dim(\mathscr{R}_\infty(t)) + \dim(\mathscr{N}_\infty(t))$ and (2) $\mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t) = \{\vec{0}\}$.

Proof Let the dimension of V be n. We will verify the second sentence, which is equivalent to the first. Clause (1) is true because any transformation satisfies that its rank plus its nullity equals the dimension of the space, and in particular this holds for the transformation t^n.

For clause (2), assume that $\vec{v} \in \mathscr{R}_\infty(t) \cap \mathscr{N}_\infty(t)$ to prove that $\vec{v} = \vec{0}$. Because $\vec{v}$ is in the generalized null space, $t^n(\vec{v}) = \vec{0}$. On the other hand, by the lemma $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one and a composition of one-to-one maps is one-to-one, so $t^n\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one. Only $\vec{0}$ is sent by a one-to-one linear map to $\vec{0}$ so the fact that $t^n(\vec{v}) = \vec{0}$ implies that $\vec{v} = \vec{0}$. QED

2.3 Remark Technically there is a difference between the map $t\colon V \to V$ and the map on the subspace $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ if the generalized range space is not equal to V, because the domains are different. But the difference is small because the second is the restriction of the first to $\mathscr{R}_\infty(t)$.

For powers between $j = 0$ and $j = n$, the space V might not be the direct sum of $\mathscr{R}(t^j)$ and $\mathscr{N}(t^j)$. The next example shows that the two can have a nontrivial intersection.

2.4 Example Consider the transformation of $\mathbb{C}^2$ defined by this action on the elements of the standard basis.

$$\begin{pmatrix}1\\0\end{pmatrix} \xmapsto{n} \begin{pmatrix}0\\1\end{pmatrix} \quad \begin{pmatrix}0\\1\end{pmatrix} \xmapsto{n} \begin{pmatrix}0\\0\end{pmatrix} \qquad N = \mathrm{Rep}_{\mathcal{E}_2,\mathcal{E}_2}(n) = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

This is a *shift map* and is clearly nilpotent of index two.

$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}0\\x\end{pmatrix}$$

Another way to depict this map's action is with a *string*.

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{0}$$

The vector

$$\vec{e}_2 = \begin{pmatrix}0\\1\end{pmatrix}$$

is in both the range space and null space.

2.5 Example A map $\hat{n}\colon \mathbb{C}^4 \to \mathbb{C}^4$ whose action on $\mathcal{E}_4$ is given by the string

$$\vec{e}_1 \mapsto \vec{e}_2 \mapsto \vec{e}_3 \mapsto \vec{e}_4 \mapsto \vec{0}$$

has $\mathscr{R}(\hat{n}) \cap \mathscr{N}(\hat{n})$ equal to the span $[\{\vec{e}_4\}]$, has $\mathscr{R}(\hat{n}^2) \cap \mathscr{N}(\hat{n}^2) = [\{\vec{e}_3, \vec{e}_4\}]$, and has $\mathscr{R}(\hat{n}^3) \cap \mathscr{N}(\hat{n}^3) = [\{\vec{e}_4\}]$. It is nilpotent of index four. The matrix representation is all zeros except for some subdiagonal ones.

$$\hat{N} = \mathrm{Rep}_{\mathcal{E}_4,\mathcal{E}_4}(\hat{n}) = \begin{pmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix}$$

2.6 Example Transformations can act via more than one string. A transformation t acting on a basis $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_5 \rangle$ by

$$\begin{array}{ccccccc} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{\beta}_3 & \mapsto & \vec{0} \\ \vec{\beta}_4 & \mapsto & \vec{\beta}_5 & \mapsto & \vec{0} \end{array}$$

is represented by a matrix that is all zeros except for blocks of subdiagonal ones

$$\mathrm{Rep}_{B,B}(t) = \left(\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

(the lines just visually organize the blocks).

In those examples all vectors are eventually transformed to zero.

2.7 Definition A *nilpotent* transformation is one with a power that is the zero map. A *nilpotent matrix* is one with a power that is the zero matrix. In either case, the least such power is the *index of nilpotency*.

2.8 Example In Example 2.4 the index of nilpotency is two. In Example 2.5 it is four. In Example 2.6 it is three.

2.9 Example The differentiation map $d/dx \colon \mathcal{P}_2 \to \mathcal{P}_2$ is nilpotent of index three since the third derivative of any quadratic polynomial is zero. This map's action is described by the string $x^2 \mapsto 2x \mapsto 2 \mapsto 0$ and taking the basis $B = \langle x^2, 2x, 2 \rangle$ gives this representation.

$$\mathrm{Rep}_{B,B}(d/dx) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Not all nilpotent matrices are all zeros except for blocks of subdiagonal ones.

2.10 Example With the matrix $\hat{N}$ from Example 2.5, and this four-vector basis

$$D = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

a change of basis operation produces this representation with respect to D, D.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -3 & -2 & 5 & 0 \\ -2 & -1 & 3 & 0 \\ 2 & 1 & -2 & 0 \end{pmatrix}$$

The new matrix is nilpotent; its fourth power is the zero matrix. We could verify this with a tedious computation or we can instead just observe that it is nilpotent since its fourth power is similar to $\hat{N}^4$, the zero matrix, and the only matrix similar to the zero matrix is itself.

$$(P\hat{N}P^{-1})^4 = P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} \cdot P\hat{N}P^{-1} = P\hat{N}^4P^{-1}$$

The goal of this subsection is to show that the prior example is prototypical in that every nilpotent matrix is similar to one that is all zeros except for blocks of subdiagonal ones.

2.11 Definition Let t be a nilpotent transformation on V. A t-*string of length* k *generated by* $\vec{v} \in V$ is a sequence $\langle \vec{v}, t(\vec{v}), \ldots, t^{k-1}(\vec{v}) \rangle$. A t-*string basis* is a basis that is a concatenation of t-strings.

2.12 Example Consider differentiation $d/dx \colon \mathcal{P}_2 \to \mathcal{P}_2$. The sequence $\langle x^2, 2x, 2, 0 \rangle$ is a d/dx-string of length 4. The sequence $\langle x^2, 2x, 2 \rangle$ is a d/dx-string of length 3 that is a basis for $\mathcal{P}_2$.

Note that the strings cannot form a basis under concatenation if they are not disjoint because a basis cannot have a repeated vector.

2.13 Example In Example 2.6, we can concatenate the t-strings $\langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$ and $\langle \vec{\beta}_4, \vec{\beta}_5 \rangle$ to make a basis for the domain of t.

2.14 Lemma If a space has a t-string basis then the index of nilpotency of t is the length of the longest string in that basis.

Proof Let the space have a basis of t-strings and let t's index of nilpotency be k. We cannot have that the longest string in that basis is longer than t's index of nilpotency because t^k sends any vector, including the vector starting the longest string, to $\vec{0}$. Therefore instead suppose that the space has a t-string basis B where all of the strings are shorter than length k. Because t has index k, there is a vector $\vec{v}$ such that $t^{k-1}(\vec{v}) \neq \vec{0}$. Represent $\vec{v}$ as a linear combination of elements from B and apply t^{k-1}. We are supposing that t^{k-1} maps each element of B to $\vec{0}$, and therefore maps each term in the linear combination to $\vec{0}$, but also that it does not map $\vec{v}$ to $\vec{0}$. That is a contradiction. QED

We shall show that each nilpotent map has an associated string basis, a basis of disjoint strings.

To see the main idea of the argument, imagine that we want to construct a counterexample, a map that is nilpotent but without an associated disjoint string basis. We might think to make something like the map $t \colon \mathbb{C}^5 \to \mathbb{C}^5$ with this action.

$$\begin{array}{l} \vec{e}_1 \\ \qquad \searrow \\ \qquad\quad \vec{e}_3 \mapsto \vec{0} \\ \qquad \nearrow \\ \vec{e}_2 \\ \\ \vec{e}_4 \mapsto \vec{e}_5 \mapsto \vec{0} \end{array} \qquad \text{Rep}_{\mathcal{E}_5,\mathcal{E}_5}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

But, the fact that the shown basis isn't disjoint doesn't mean that there isn't another basis that consists of disjoint strings.

To produce such a basis for this map we will first find the number and lengths of its strings. Observer that t's index of nilpotency is two. Lemma 2.14 says that at least one string in a disjoint string basis has length two. There are five basis elements so if there is a disjoint string basis then the map must act in one of these ways.

$$\begin{array}{ccccc} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{0} \\ \vec{\beta}_3 & \mapsto & \vec{\beta}_4 & \mapsto & \vec{0} \\ \vec{\beta}_5 & \mapsto & \vec{0} & & \end{array} \qquad \begin{array}{ccccc} \vec{\beta}_1 & \mapsto & \vec{\beta}_2 & \mapsto & \vec{0} \\ \vec{\beta}_3 & \mapsto & \vec{0} & & \\ \vec{\beta}_4 & \mapsto & \vec{0} & & \\ \vec{\beta}_5 & \mapsto & \vec{0} & & \end{array}$$

Now, the key point. A transformation with the left-hand action has a null space of dimension three since that's how many basis vectors are mapped to zero. A transformation with the right-hand action has a null space of dimension four. Wit the matrix representation above we can determine which of the two possible shapes is right.

$$\mathscr{N}(t) = \{ \begin{pmatrix} x \\ -x \\ z \\ 0 \\ r \end{pmatrix} \mid x, z, r \in \mathbb{C} \}$$

This is three-dimensional, meaning that of the two disjoint string basis forms above, t's basis has the left-hand one.

To produce a string basis for t, first pick $\vec{\beta}_2$ and $\vec{\beta}_4$ from $\mathscr{R}(t) \cap \mathscr{N}(t)$.

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

(Other choices are possible, just be sure that the set $\{\vec{\beta}_2, \vec{\beta}_4\}$ is linearly independent.) For $\vec{\beta}_5$ pick a vector from $\mathscr{N}(t)$ that is not in the span of $\{\vec{\beta}_2, \vec{\beta}_4\}$.

$$\vec{\beta}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Finally, take $\vec{\beta}_1$ and $\vec{\beta}_3$ such that $t(\vec{\beta}_1) = \vec{\beta}_2$ and $t(\vec{\beta}_3) = \vec{\beta}_4$.

$$\vec{\beta}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, we have a string basis $B = \langle \vec{\beta}_1, \dots, \vec{\beta}_5 \rangle$ and with respect to that basis the matrix of t has blocks of subdiagonal 1's.

$$\text{Rep}_{B,B}(t) = \left(\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

2.15 Theorem Any nilpotent transformation t is associated with a t-string basis. While the basis is not unique, the number and the length of the strings is determined by t.

This illustrates the proof, which describes three kinds of basis vectors (shown in squares if they are in the null space and in circles if they are not).

$$\begin{array}{cccccccccccccc}
3 & \mapsto & 1 & \mapsto & \cdots & & & & \cdots & \mapsto & 1 & \mapsto & \boxed{1} & \mapsto \vec{0} \\
3 & \mapsto & 1 & \mapsto & \cdots & & & & \cdots & \mapsto & 1 & \mapsto & \boxed{1} & \mapsto \vec{0} \\
 & \vdots & & & & & & & & & & & & \\
3 & \mapsto & 1 & \mapsto & \cdots & \mapsto & 1 & \mapsto & \boxed{1} & \mapsto & \vec{0} & & & \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & & & & & \\
 & \vdots & & & & & & & & & & & & \\
\boxed{2} & \mapsto & \vec{0} & & & & & & & & & & &
\end{array}$$

Proof Fix a vector space V. We will argue by induction on the index of nilpotency. If the map $t\colon V \to V$ has index of nilpotency 1 then it is the zero map and any basis is a string basis $\vec{\beta}_1 \mapsto \vec{0}, \dots, \vec{\beta}_n \mapsto \vec{0}$.

For the inductive step, assume that the theorem holds for any transformation $t\colon V \to V$ with an index of nilpotency between 1 and $k-1$ (with $k>1$) and consider the index k case.

Observe that the restriction of t to the range space $t\colon \mathscr{R}(t) \to \mathscr{R}(t)$ is also nilpotent, of index $k-1$. Apply the inductive hypothesis to get a string basis for $\mathscr{R}(t)$, where the number and length of the strings is determined by t.

$$B = \langle \vec{\beta}_1, t(\vec{\beta}_1), \ldots, t^{h_1}(\vec{\beta}_1) \rangle ^\frown \langle \vec{\beta}_2, \ldots, t^{h_2}(\vec{\beta}_2) \rangle ^\frown \cdots ^\frown \langle \vec{\beta}_i, \ldots, t^{h_i}(\vec{\beta}_i) \rangle$$

(In the illustration above these are the vectors of kind 1.)

Note that taking the final nonzero vector in each of these strings gives a basis $C = \langle t^{h_1}(\vec{\beta}_1), \ldots, t^{h_i}(\vec{\beta}_i) \rangle$ for the intersection $\mathscr{R}(t) \cap \mathscr{N}(t)$. This is because a member of $\mathscr{R}(t)$ maps to zero if and only if it is a linear combination of those basis vectors that map to zero. (The illustration shows these as 1's in squares.)

Now extend C to a basis for all of $\mathscr{N}(t)$.

$$\hat{C} = C ^\frown \langle \vec{\xi}_1, \ldots, \vec{\xi}_p \rangle$$

(In the illustration the $\vec{\xi}$'s are the vectors of kind 2 and so the set $\hat{C}$ is the set of vectors in squares.) While the vectors $\vec{\xi}$ we choose aren't uniquely determined by t, what is uniquely determined is the number of them: it is the dimension of $\mathscr{N}(t)$ minus the dimension of $\mathscr{R}(t) \cap \mathscr{N}(t)$.

Finally, $B ^\frown \hat{C}$ is a basis for $\mathscr{R}(t) + \mathscr{N}(t)$ because any sum of something in the range space with something in the null space can be represented using elements of B for the range space part and elements of $\hat{C}$ for the part from the null space. Note that

$$\begin{aligned} \dim\big(\mathscr{R}(t) + \mathscr{N}(t)\big) &= \dim(\mathscr{R}(t)) + \dim(\mathscr{N}(t)) - \dim(\mathscr{R}(t) \cap \mathscr{N}(t)) \\ &= \operatorname{rank}(t) + \operatorname{nullity}(t) - i \\ &= \dim(V) - i \end{aligned}$$

and so we can extend $B ^\frown \hat{C}$ to a basis for all of V by the addition of i more vectors, provided that they are not linearly dependent on what we have already. Recall that each of $\vec{\beta}_1, \ldots, \vec{\beta}_i$ is in $\mathscr{R}(t)$, and extend $B ^\frown \hat{C}$ with vectors $\vec{v}_1, \ldots, \vec{v}_i$ such that $t(\vec{v}_1) = \vec{\beta}_1, \ldots, t(\vec{v}_i) = \vec{\beta}_i$. (In the illustration these are the 3's.) The check that this extension preserves linear independence is Exercise 31. QED

2.16 Corollary Every nilpotent matrix is similar to a matrix that is all zeros except for blocks of subdiagonal ones. That is, every nilpotent map is represented with respect to some basis by such a matrix.

This form is unique in the sense that if a nilpotent matrix is similar to two such matrices then those two simply have their blocks ordered differently. Thus this is a canonical form for the similarity classes of nilpotent matrices provided that we order the blocks, say, from longest to shortest.

2.17 Example The matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

has an index of nilpotency of two, as this calculation shows.

power p	M^p	$\mathscr{N}(M^p)$
1	$M = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$	$\{\begin{pmatrix} x \\ x \end{pmatrix} \mid x \in \mathbb{C}\}$
2	$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\mathbb{C}^2$

Because the matrix is 2×2, any transformation that it represents is on a space of dimension two. The nullspace of one application of the map $\mathscr{N}(m)$ has dimension one, and the nullspace of two applications $\mathscr{N}(m^2)$ has dimension two. Thus the action of m on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$ and the canonical form of the matrix is this.

$$N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We can exhibit such a string basis, and also the change of basis matrices witnessing the matrix similarity between M and N. Suppose that $m\colon \mathbb{C}^2 \to \mathbb{C}^2$ is such that M represents it with respect to the standard bases. (We could take M to be a representation with respect to some other basis but the standard one is convenient.) Pick $\vec{\beta}_2 \in \mathscr{N}(m)$. Also pick $\vec{\beta}_1$ so that $m(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For the change of basis matrices, recall the similarity diagram.

$$\begin{array}{ccc} \mathbb{C}^2_{wrt\ \mathcal{E}_2} & \xrightarrow[M]{m} & \mathbb{C}^2_{wrt\ \mathcal{E}_2} \\ \text{id}\big\downarrow P & & \text{id}\big\downarrow P \\ \mathbb{C}^2_{wrt\ B} & \xrightarrow[N]{m} & \mathbb{C}^2_{wrt\ B} \end{array}$$

The canonical form equals $\text{Rep}_{B,B}(m) = PMP^{-1}$, where

$$P^{-1} = \text{Rep}_{B,\mathcal{E}_2}(\text{id}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad P = (P^{-1})^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and the verification of the matrix calculation is routine.

$$\begin{pmatrix}1&-1\\0&1\end{pmatrix}\begin{pmatrix}1&-1\\1&-1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

2.18 Example This matrix

$$\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\-1&1&1&-1&1\\0&1&0&0&0\\1&0&-1&1&-1\end{pmatrix}$$

is nilpotent, of index 3.

power p	N^p	$\mathscr{N}(N^p)$
1	$\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\-1&1&1&-1&1\\0&1&0&0&0\\1&0&-1&1&-1\end{pmatrix}$	$\{\begin{pmatrix}0\\0\\u-v\\u\\v\end{pmatrix}\mid u,v\in\mathbb{C}\}$
2	$\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\end{pmatrix}$	$\{\begin{pmatrix}0\\y\\z\\u\\v\end{pmatrix}\mid y,z,u,v\in\mathbb{C}\}$
3	*–zero matrix–*	$\mathbb{C}^5$

The table tells us this about any string basis: the null space after one map application has dimension two so two basis vectors map directly to zero, the null space after the second application has dimension four so two additional basis vectors map to zero by the second iteration, and the null space after three applications is of dimension five so the remaining one basis vector maps to zero in three hops.

$$\begin{array}{l}\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{\beta}_3\mapsto\vec{0}\\ \vec{\beta}_4\mapsto\vec{\beta}_5\mapsto\vec{0}\end{array}$$

To produce such a basis, first pick two vectors from $\mathscr{N}(n)$ that form a linearly independent set.

$$\vec{\beta}_3=\begin{pmatrix}0\\0\\1\\1\\0\end{pmatrix}\quad\vec{\beta}_5=\begin{pmatrix}0\\0\\0\\1\\1\end{pmatrix}$$

Then add $\vec{\beta}_2, \vec{\beta}_4 \in \mathscr{N}(n^2)$ such that $n(\vec{\beta}_2) = \vec{\beta}_3$ and $n(\vec{\beta}_4) = \vec{\beta}_5$.

$$\vec{\beta}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\beta}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Finish by adding $\vec{\beta}_1$ such that $n(\vec{\beta}_1) = \vec{\beta}_2$.

$$\vec{\beta}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Exercises

✓ 2.19 What is the index of nilpotency of the *right-shift* operator, here acting on the space of triples of reals?

$$(x, y, z) \mapsto (0, x, y)$$

✓ 2.20 For each string basis state the index of nilpotency and give the dimension of the range space and null space of each iteration of the nilpotent map.

(a) $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$
$\vec{\beta}_3 \mapsto \vec{\beta}_4 \mapsto \vec{0}$

(b) $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$
$\vec{\beta}_4 \mapsto \vec{0}$
$\vec{\beta}_5 \mapsto \vec{0}$
$\vec{\beta}_6 \mapsto \vec{0}$

(c) $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$

Also give the canonical form of the matrix.

2.21 Decide which of these matrices are nilpotent.

(a) $\begin{pmatrix} -2 & 4 \\ -1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} -3 & 2 & 1 \\ -3 & 2 & 1 \\ -3 & 2 & 1 \end{pmatrix}$ **(d)** $\begin{pmatrix} 1 & 1 & 4 \\ 3 & 0 & -1 \\ 5 & 2 & 7 \end{pmatrix}$

(e) $\begin{pmatrix} 45 & -22 & -19 \\ 33 & -16 & -14 \\ 69 & -34 & -29 \end{pmatrix}$

✓ 2.22 Find the canonical form of this matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

✓ 2.23 Consider the matrix from Example 2.18.

(a) Use the action of the map on the string basis to give the canonical form.

(b) Find the change of basis matrices that bring the matrix to canonical form.

(c) Use the answer in the prior item to check the answer in the first item.

✓ 2.24 Each of these matrices is nilpotent.

(a) $\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

Put each in canonical form.

2.25 Describe the effect of left or right multiplication by a matrix that is in the canonical form for nilpotent matrices.

2.26 Is nilpotence invariant under similarity? That is, must a matrix similar to a nilpotent matrix also be nilpotent? If so, with the same index?

✓ 2.27 Show that the only eigenvalue of a nilpotent matrix is zero.

2.28 Is there a nilpotent transformation of index three on a two-dimensional space?

2.29 In the proof of Theorem 2.15, why isn't the proof's base case that the index of nilpotency is zero?

✓ 2.30 Let $t\colon V \to V$ be a linear transformation and suppose $\vec{v} \in V$ is such that $t^k(\vec{v}) = \vec{0}$ but $t^{k-1}(\vec{v}) \neq \vec{0}$. Consider the t-string $\langle \vec{v}, t(\vec{v}), \ldots, t^{k-1}(\vec{v}) \rangle$.

(a) Prove that t is a transformation on the span of the set of vectors in the string, that is, prove that t restricted to the span has a range that is a subset of the span. We say that the span is a t-*invariant* subspace.

(b) Prove that the restriction is nilpotent.

(c) Prove that the t-string is linearly independent and so is a basis for its span.

(d) Represent the restriction map with respect to the t-string basis.

2.31 Finish the proof of Theorem 2.15.

2.32 Show that the terms 'nilpotent transformation' and 'nilpotent matrix', as given in Definition 2.7, fit with each other: a map is nilpotent if and only if it is represented by a nilpotent matrix. (Is it that a transformation is nilpotent if an only if there is a basis such that the map's representation with respect to that basis is a nilpotent matrix, or that any representation is a nilpotent matrix?)

2.33 Let T be nilpotent of index four. How big can the range space of T^3 be?

2.34 Recall that similar matrices have the same eigenvalues. Show that the converse does not hold.

2.35 Lemma 2.1 shows that any for any linear transformation $t\colon V \to V$ the restriction $t\colon \mathscr{R}_\infty(t) \to \mathscr{R}_\infty(t)$ is one-to-one. Show that it is also onto, so it is an automorphism. Must it be the identity map?

2.36 Prove that a nilpotent matrix is similar to one that is all zeros except for blocks of super-diagonal ones.

✓ 2.37 Prove that if a transformation has the same range space as null space. then the dimension of its domain is even.

2.38 Prove that if two nilpotent matrices commute then their product and sum are also nilpotent.

2.39 Consider the transformation of $\mathcal{M}_{n\times n}$ given by $t_S(T) = ST - TS$ where S is an $n\times n$ matrix. Prove that if S is nilpotent then so is t_S.

2.40 Show that if N is nilpotent then $I - N$ is invertible. Is that 'only if' also?

IV Jordan Form

This section uses material from three optional subsections: Combining Subspaces, Determinants Exist, and Laplace's Expansion.

We began this chapter by remembering that every linear map $h\colon V \to W$ can be represented by a partial identity matrix with respect to some bases $B \subset V$ and $D \subset W$. That is, the partial identity form is a canonical form for matrix equivalence. This chapter considers transformations, where the codomain equals the domain, so we naturally ask what is possible when the two bases are equal $\mathrm{Rep}_{B,B}(t)$. In short, we want a canonical form for matrix similarity.

We noted that in the B, B case a partial identity matrix is not always possible. We therefore extended the matrix forms of interest to the natural generalization, diagonal matrices, and showed that a transformation or square matrix can be diagonalized if its eigenvalues are distinct. But at the same time we gave an example of a square matrix that cannot be diagonalized (because it is nilpotent) and thus diagonal form won't suffice as the canonical form for matrix similarity.

The prior section developed that example to get a canonical form, subdiagonal ones, for nilpotent matrices.

This section finishes our program by showing that for any linear transformation there is a basis such that the matrix representation $\mathrm{Rep}_{B,B}(t)$ is the sum of a diagonal matrix and a nilpotent matrix. This is Jordan canonical form.

IV.1 Polynomials of Maps and Matrices

Recall that the set of square matrices $\mathcal{M}_{n\times n}$ is a vector space under entry-by-entry addition and scalar multiplication, and that this space has dimension n^2. Thus, for any $n\times n$ matrix T the n^2+1-member set $\{I, T, T^2, \dots, T^{n^2}\}$ is linearly dependent and so there are scalars $c_0, \dots, c_{n^2}$, not all zero, such that

$$c_{n^2}T^{n^2} + \cdots + c_1T + c_0I$$

is the zero matrix. Therefore every transformation has a kind of generalized nilpotency: the powers of a square matrix cannot climb forever without a "repeat."

1.1 Example Rotation of plane vectors $\pi/6$ radians counterclockwise is represented with respect to the standard basis by

$$T = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

and verifying that $0T^4 + 0T^3 + 1T^2 - 2T - 1I$ equals the zero matrix is easy.

1.2 Definition Let t be a linear transformation of a vector space V. Where $f(x) = c_n x^n + \cdots + c_1 x + c_0$ is a polynomial, $f(t)$ is the transformation $c_n t^n + \cdots + c_1 t + c_0(\text{id})$ on V. In the same way, if T is a square matrix then $f(T)$ is the matrix $c_n T^n + \cdots + c_1 T + c_0 I$.

The polynomial of the matrix represents the polynomial of the map: if $T = \text{Rep}_{B,B}(t)$ then $f(T) = \text{Rep}_{B,B}(f(t))$. This is because $T^j = \text{Rep}_{B,B}(t^j)$, and $cT = \text{Rep}_{B,B}(ct)$, and $T_1 + T_2 = \text{Rep}_{B,B}(t_1 + t_2)$.

1.3 Remark Most authors write the matrix polynomial slightly differently than the map polynomial. For instance, if $f(x) = x - 3$ then most authors explicitly write the identity matrix $f(T) = T - 3I$ but don't write the identity map $f(t) = t - 3$. We shall follow this convention.

Consider again Example 1.1. The space $\mathcal{M}_{2\times2}$ has dimension four so we know that for any 2×2 matrix there is a fourth degree polynomial f such that $f(T)$ equals the zero matrix. But for the T in that example we exhibited a polynomial of degree less than four that gives the zero matrix. So while degree n^2 always suffices, in some cases a smaller-degree polynomial works.

1.4 Definition The *minimal polynomial* $m(x)$ of a transformation t or a square matrix T is the polynomial of least degree and with leading coefficient one such that $m(t)$ is the zero map or $m(T)$ is the zero matrix.

A minimal polynomial cannot be the zero polynomial because of the restriction on the leading coefficient. Obviously no other constant polynomial would do, so a minimal polynomial must have degree at least one. Thus, the zero matrix has minimal polynomial $p(x) = x$ while the identity matrix has minimal polynomial $\hat{p}(x) = x - 1$.

1.5 Lemma Any transformation or square matrix has a unique minimal polynomial.

Proof We first prove existence. By the earlier observation that degree n^2 suffices, there is at least one polynomial $p(x) = c_k x^k + \cdots + c_0$ that takes the map or matrix to zero, and it is not the zero polynomial by the prior paragraph. From among all such polynomials there must be at least one with minimal degree. Divide this polynomial by its leading coefficient c_k to get a leading 1. Hence any map or matrix has a minimal polynomial.

Now for uniqueness. Suppose that $m(x)$ and $\hat{m}(x)$ both take the map or matrix to zero, are both of minimal degree and are thus of equal degree, and both have a leading 1. Subtract: $d(x) = m(x) - \hat{m}(x)$. This polynomial takes the map or matrix to zero and since the leading terms of m and $\hat{m}$ cancel, d is of smaller degree than the other two. If d were to have a nonzero leading coefficient then we could divide by it to get a polynomial that takes the map or matrix to zero and has leading coefficient 1. This would contradict the minimality of the degree of m and $\hat{m}$. Thus the leading coefficient of d is zero, so $m(x)-\hat{m}(x)$ is the zero polynomial, and so the two are equal. QED

1.6 Example We can compute that $m(x)=x^2-2x-1$ is minimal for the matrix of Example 1.1 by finding the powers of T up to $n^2=4$.

$$T^2=\begin{pmatrix}1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & 1/2\end{pmatrix}\quad T^3=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\quad T^4=\begin{pmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{pmatrix}$$

Put $c_4T^4+c_3T^3+c_2T^2+c_1T+c_0I$ equal to the zero matrix

$$\begin{array}{rcl}
-(1/2)c_4 \qquad\quad + \;\; (1/2)c_2+(\sqrt{3}/2)c_1+c_0 &=& 0\\
-(\sqrt{3}/2)c_4-c_3-(\sqrt{3}/2)c_2-\;\;(1/2)c_1 \qquad\quad &=& 0\\
(\sqrt{3}/2)c_4+c_3+(\sqrt{3}/2)c_2+\;\;(1/2)c_1 \qquad\quad &=& 0\\
-(1/2)c_4 \qquad\quad + \;\; (1/2)c_2+(\sqrt{3}/2)c_1+c_0 &=& 0
\end{array}$$

and use Gauss' Method.

$$\begin{array}{rcl}
c_4 \quad - \quad c_2-\sqrt{3}c_1-\;\;2c_0 &=& 0\\
c_3+\sqrt{3}c_2+\;\;2c_1+\sqrt{3}c_0 &=& 0
\end{array}$$

Setting c_4, c_3, and c_2 to zero forces c_1 and c_0 to also come out as zero. To get a leading one, the most we can do is to set c_4 and c_3 to zero. Thus the minimal polynomial is quadratic.

Using the method of that example to find the minimal polynomial of a 3×3 matrix would mean doing Gaussian reduction on a system with nine equations in ten unknowns. We shall develop an alternative.

1.7 Lemma Suppose that the polynomial $f(x)=c_nx^n+\cdots+c_1x+c_0$ factors as $k(x-\lambda_1)^{q_1}\cdots(x-\lambda_z)^{q_z}$. If t is a linear transformation then these two are equal maps.

$$c_nt^n+\cdots+c_1t+c_0=k\cdot(t-\lambda_1)^{q_1}\circ\cdots\circ(t-\lambda_z)^{q_z}$$

Consequently, if T is a square matrix then $f(T)$ and $k\cdot(T-\lambda_1I)^{q_1}\cdots(T-\lambda_zI)^{q_z}$ are equal matrices.

Proof We use induction on the degree of the polynomial. The cases where the polynomial is of degree zero and degree one are clear. The full induction argument is Exercise 1.7 but we will give its sense with the degree two case.

A quadratic polynomial factors into two linear terms $f(x) = k(x - \lambda_1) \cdot (x - \lambda_2) = k(x^2 + (-\lambda_1 - \lambda_2)x + \lambda_1\lambda_2)$ (the roots λ_1 and λ_2 could be equal). We can check that substituting t for x in the factored and unfactored versions gives the same map.

$$\begin{aligned}\big(k \cdot (t - \lambda_1) \circ (t - \lambda_2)\big)\,(\vec{v}) &= \big(k \cdot (t - \lambda_1)\big)\,(t(\vec{v}) - \lambda_2\vec{v}) \\ &= k \cdot \big(t(t(\vec{v})) - t(\lambda_2\vec{v}) - \lambda_1 t(\vec{v}) - \lambda_1\lambda_2\vec{v}\big) \\ &= k \cdot \big(t \circ t\,(\vec{v}) - (\lambda_1 + \lambda_2)t(\vec{v}) + \lambda_1\lambda_2\vec{v}\big) \\ &= k \cdot (t^2 - (\lambda_1 + \lambda_2)t + \lambda_1\lambda_2)\,(\vec{v})\end{aligned}$$

The third equality holds because the scalar λ_2 comes out of the second term, since t is linear. QED

In particular, if a minimal polynomial $m(x)$ for a transformation t factors as $m(x) = (x - \lambda_1)^{q_1} \cdots (x - \lambda_z)^{q_z}$ then $m(t) = (t - \lambda_1)^{q_1} \circ \cdots \circ (t - \lambda_z)^{q_z}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t - \lambda_i$ sends some nonzero vectors to zero. Exactly the same holds in the matrix case — if m is minimal for T then $m(T) = (T - \lambda_1 I)^{q_1} \cdots (T - \lambda_z I)^{q_z}$ is the zero matrix and at least one of the matrices $T - \lambda_i I$ sends some nonzero vectors to zero. That is, in both cases at least some of the λ_i are eigenvalues. (Exercise 29 expands on this.)

The next result is that every root of the minimal polynomial is an eigenvalue, and further that every eigenvalue is a root of the minimal polynomial (i.e, below it says '$1 \leqslant q_i$' and not just '$0 \leqslant q_i$'). For that result, recall that to find eigenvalues we solve $|T - xI| = 0$ and this determinant gives a polynomial in x, called the characteristic polynomial, whose roots are the eigenvalues.

1.8 Theorem (Cayley-Hamilton) If the characteristic polynomial of a transformation or square matrix factors into

$$k \cdot (x - \lambda_1)^{p_1}(x - \lambda_2)^{p_2} \cdots (x - \lambda_z)^{p_z}$$

then its minimal polynomial factors into

$$(x - \lambda_1)^{q_1}(x - \lambda_2)^{q_2} \cdots (x - \lambda_z)^{q_z}$$

where $1 \leqslant q_i \leqslant p_i$ for each i between 1 and z.

The proof takes up the next three lemmas. We will state them in matrix terms but they apply equally well to maps. (The matrix version is convenient for the first proof.)

The first result is the key. For the proof, observe that we can view a matrix of polynomials as a polynomial with matrix coefficients.

$$\begin{pmatrix} 2x^2+3x-1 & x^2+2 \\ 3x^2+4x+1 & 4x^2+x+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} x^2 + \begin{pmatrix} 3 & 0 \\ 4 & 1 \end{pmatrix} x + \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$$

1.9 Lemma If T is a square matrix with characteristic polynomial $c(x)$ then $c(T)$ is the zero matrix.

PROOF Let C be $T - xI$, the matrix whose determinant is the characteristic polynomial $c(x) = c_n x^n + \cdots + c_1 x + c_0$.

$$C = \begin{pmatrix} t_{1,1} - x & t_{1,2} & \ldots & \\ t_{2,1} & t_{2,2} - x & & \\ \vdots & & \ddots & \\ & & & t_{n,n} - x \end{pmatrix}$$

Recall Theorem Four.III.1.9, that the product of a matrix with its adjoint equals the determinant of the matrix times the identity.

$$c(x) \cdot I = \operatorname{adj}(C)C = \operatorname{adj}(C)(T - xI) = \operatorname{adj}(C)T - \operatorname{adj}(C) \cdot x \qquad (*)$$

The left side of $(*)$ is $c_n I x^n + c_{n-1} I x^{n-1} + \cdots + c_1 I x + c_0 I$. For the right side, the entries of $\operatorname{adj}(C)$ are polynomials, each of degree at most $n-1$ since the minors of a matrix drop a row and column. As suggested before the proof, rewrite it as a polynomial with matrix coefficients: $\operatorname{adj}(C) = C_{n-1}x^{n-1} + \cdots + C_1 x + C_0$ where each C_i is a matrix of scalars. Now this is the right side of $(*)$.

$$[(C_{n-1}T)x^{n-1} + \cdots + (C_1 T)x + C_0 T] - [C_{n-1}x^n - C_{n-2}x^{n-1} - \cdots - C_0 x]$$

Equate the left and right side of $(*)$'s coefficients of x^n, of x^{n-1}, etc.

$$\begin{aligned} c_n I &= -C_{n-1} \\ c_{n-1} I &= -C_{n-2} + C_{n-1}T \\ &\vdots \\ c_1 I &= -C_0 + C_1 T \\ c_0 I &= C_0 T \end{aligned}$$

Multiply, from the right, both sides of the first equation by T^n, both sides of the second equation by T^{n-1}, etc.

$$\begin{aligned}
c_n T^n &= -C_{n-1} T^n \\
c_{n-1} T^{n-1} &= -C_{n-2} T^{n-1} + C_{n-1} T^n \\
&\vdots \\
c_1 T &= -C_0 T + C_1 T^2 \\
c_0 I &= C_0 T
\end{aligned}$$

Add. The left is $c_n T^n + c_{n-1} T^{n-1} + \cdots + c_0 I$. The right telescopes; for instance $-C_{n-1} T^n$ from the first line combines with the $C_{n-1} T^n$ half of the second line. The total on the right is the zero matrix. QED

We refer to that result by saying that a matrix or map *satisfies* its characteristic polynomial.

1.10 Lemma Where $f(x)$ is a polynomial, if $f(T)$ is the zero matrix then $f(x)$ is divisible by the minimal polynomial of T. That is, any polynomial that is satisfied by T is divisible by T's minimal polynomial.

PROOF Let $m(x)$ be minimal for T. The Division Theorem for Polynomials gives $f(x) = q(x)m(x) + r(x)$ where the degree of r is strictly less than the degree of m. Because T satisfies both f and m, plugging T into that equation gives that $r(T)$ is the zero matrix. That contradicts the minimality of m unless r is the zero polynomial. QED

Combining the prior two lemmas shows that the minimal polynomial divides the characteristic polynomial. Thus any root of the minimal polynomial is also a root of the characteristic polynomial. That is, so far we have that if $m(x) = (x-\lambda_1)^{q_1} \cdots (x-\lambda_i)^{q_i}$ then $c(x)$ has the form $(x-\lambda_1)^{p_1} \cdots (x-\lambda_i)^{p_i}(x-\lambda_{i+1})^{p_{i+1}} \cdots (x-\lambda_z)^{p_z}$ where each q_j is less than or equal to p_j. We finish the proof of the Cayley-Hamilton Theorem by showing that the characteristic polynomial has no additional roots, that is, there are no λ_{i+1}, λ_{i+2}, etc.

1.11 Lemma Each linear factor of the characteristic polynomial of a square matrix is also a linear factor of the minimal polynomial.

PROOF Let T be a square matrix with minimal polynomial $m(x)$ and assume that $x-\lambda$ is a factor of the characteristic polynomial of T, that λ is an eigenvalue of T. We must show that $x-\lambda$ is a factor of m, i.e., that $m(\lambda) = 0$.

Suppose that λ is an eigenvalue of T with associated eigenvector $\vec{v}$. Then $T \cdot T\vec{v} = T \cdot \lambda\vec{v} = \lambda T\vec{v} = \lambda^2\vec{v}$. Similarly, $T^n\vec{v} = \lambda^n\vec{v}$. With that, we have that for any polynomial function $p(x)$, application of the matrix $p(T)$ to $\vec{v}$ equals the result of multiplying $\vec{v}$ by the scalar $p(\lambda)$.

$$\begin{aligned} p(T)\cdot\vec{v} = (c_kT^k+\cdots+c_1T+c_0I)\cdot\vec{v} &= c_kT^k\vec{v}+\cdots+c_1T\vec{v}+c_0\vec{v} \\ &= c_k\lambda^k\vec{v}+\cdots+c_1\lambda\vec{v}+c_0\vec{v}=p(\lambda)\cdot\vec{v} \end{aligned}$$

Since $m(T)$ is the zero matrix, $\vec{0}=m(T)(\vec{v})=m(\lambda)\cdot\vec{v}$ for all $\vec{v}$, and hence $m(\lambda)=0$. QED

That concludes the proof of the Cayley-Hamilton Theorem.

1.12 Example We can use the Cayley-Hamilton Theorem to find the minimal polynomial of this matrix.

$$T=\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

First we find its characteristic polynomial $c(x)=(x-1)(x-2)^3$ with the usual determinant. Now, the Cayley-Hamilton Theorem says that T's minimal polynomial is either $(x-1)(x-2)$ or $(x-1)(x-2)^2$ or $(x-1)(x-2)^3$. We can decide among the choices just by computing

$$(T-1I)(T-2I)=\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$(T-1I)(T-2I)^2=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so $m(x)=(x-1)(x-2)^2$.

Exercises

✓ 1.13 What are the possible minimal polynomials if a matrix has the given characteristic polynomial?

(a) $(x-3)^4$ **(b)** $(x+1)^3(x-4)$ **(c)** $(x-2)^2(x-5)^2$
(d) $(x+3)^2(x-1)(x-2)^2$

What is the degree of each possibility?

✓ 1.14 Find the minimal polynomial of each matrix.

(a) $\begin{pmatrix}3&0&0\\1&3&0\\0&0&4\end{pmatrix}$ (b) $\begin{pmatrix}3&0&0\\1&3&0\\0&0&3\end{pmatrix}$ (c) $\begin{pmatrix}3&0&0\\1&3&0\\0&1&3\end{pmatrix}$ (d) $\begin{pmatrix}2&0&1\\0&6&2\\0&0&2\end{pmatrix}$

(e) $\begin{pmatrix}2&2&1\\0&6&2\\0&0&2\end{pmatrix}$ (f) $\begin{pmatrix}-1&4&0&0&0\\0&3&0&0&0\\0&-4&-1&0&0\\3&-9&-4&2&-1\\1&5&4&1&4\end{pmatrix}$

1.15 Find the minimal polynomial of this matrix.

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$

✓ 1.16 What is the minimal polynomial of the differentiation operator d/dx on $\mathcal{P}_n$?

✓ 1.17 Find the minimal polynomial of matrices of this form

$$\begin{pmatrix}\lambda&0&0&\ldots&&0\\1&\lambda&0&&&0\\0&1&\lambda&&&\\&&&\ddots&&\\&&&&\lambda&0\\0&0&\ldots&&1&\lambda\end{pmatrix}$$

where the scalar λ is fixed (i.e., is not a variable).

1.18 What is the minimal polynomial of the transformation of $\mathcal{P}_n$ that sends $p(x)$ to $p(x+1)$?

1.19 What is the minimal polynomial of the map $\pi\colon \mathbb{C}^3 \to \mathbb{C}^3$ projecting onto the first two coordinates?

1.20 Find a 3×3 matrix whose minimal polynomial is x^2.

1.21 What is wrong with this claimed proof of Lemma 1.9: "if $c(x) = |T - xI|$ then $c(T) = |T - TI| = 0$"? [Cullen]

1.22 Verify Lemma 1.9 for 2×2 matrices by direct calculation.

✓ 1.23 Prove that the minimal polynomial of an $n\times n$ matrix has degree at most n (not n^2 as a person might guess from this subsection's opening). Verify that this maximum, n, can happen.

✓ 1.24 Show that, on a nontrivial vector space, a linear transformation is nilpotent if and only if its only eigenvalue is zero.

1.25 What is the minimal polynomial of a zero map or matrix? Of an identity map or matrix?

✓ 1.26 Interpret the minimal polynomial of Example 1.1 geometrically.

1.27 What is the minimal polynomial of a diagonal matrix?

✓ 1.28 A *projection* is any transformation t such that $t^2 = t$. (For instance, consider the transformation of the plane $\mathbb{R}^2$ projecting each vector onto its first coordinate. If we project twice then we get the same result as if we project just once.) What is the minimal polynomial of a projection?

1.29 *The first two items of this question are review.*

(a) Prove that the composition of one-to-one maps is one-to-one.

(b) Prove that if a linear map is not one-to-one then at least one nonzero vector from the domain maps to the zero vector in the codomain.

(c) Verify the statement, excerpted here, that precedes Theorem 1.8.

> ... if a minimal polynomial $m(x)$ for a transformation t factors as $m(x) = (x-\lambda_1)^{q_1} \cdots (x-\lambda_z)^{q_z}$ then $m(t) = (t-\lambda_1)^{q_1} \circ \cdots \circ (t-\lambda_z)^{q_z}$ is the zero map. Since $m(t)$ sends every vector to zero, at least one of the maps $t-\lambda_i$ sends some nonzero vectors to zero. ... That is, ... at least some of the λ_i are eigenvalues.

1.30 True or false: for a transformation on an n dimensional space, if the minimal polynomial has degree n then the map is diagonalizable.

1.31 Let $f(x)$ be a polynomial. Prove that if A and B are similar matrices then $f(A)$ is similar to $f(B)$.

(a) Now show that similar matrices have the same characteristic polynomial.

(b) Show that similar matrices have the same minimal polynomial.

(c) Decide if these are similar.

$$\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}$$

1.32 **(a)** Show that a matrix is invertible if and only if the constant term in its minimal polynomial is not 0.

(b) Show that if a square matrix T is not invertible then there is a nonzero matrix S such that ST and TS both equal the zero matrix.

✓ 1.33 **(a)** Finish the proof of Lemma 1.7.

(b) Give an example to show that the result does not hold if t is not linear.

1.34 Any transformation or square matrix has a minimal polynomial. Does the converse hold?

IV.2 Jordan Canonical Form

We are looking for a canonical form for matrix similarity. This subsection completes this program by moving from the canonical form for the classes of nilpotent matrices to the canonical form for all classes.

2.1 Lemma A linear transformation on a nontrivial vector space is nilpotent if and only if its only eigenvalue is zero.

Proof Let the linear transformation be $t\colon V \to V$. If t is nilpotent then there is an n such that t^n is the zero map, so t satisfies the polynomial $p(x) = x^n = (x-0)^n$. By Lemma 1.10 the minimal polynomial of t divides p, so the minimal polynomial has only zero for a root. By Cayley-Hamilton, Theorem 1.8, the characteristic polynomial has only zero for a root. Thus the only eigenvalue of t is zero.

Conversely, if a transformation t on an n-dimensional space has only the single eigenvalue of zero then its characteristic polynomial is x^n. Lemma 1.9 says that a map satisfies its characteristic polynomial so t^n is the zero map. Thus t is nilpotent. QED

The 'nontrivial vector space' is in the statement of that lemma because on a trivial space $\{\vec{0}\}$ the only transformation is the zero map, which has no eigenvalues because there are no associated nonzero eigenvectors.

2.2 Corollary The transformation $t-\lambda$ is nilpotent if and only if t's only eigenvalue is λ.

Proof The transformation $t-\lambda$ is nilpotent if and only if $t-\lambda$'s only eigenvalue is 0. That holds if and only if t's only eigenvalue is λ, because $t(\vec{v}) = \lambda\vec{v}$ if and only if $(t-\lambda)\,(\vec{v}) = 0 \cdot \vec{v}$. QED

We already have the canonical form that we want for the case of nilpotent matrices, that is, for each matrix whose only eigenvalue is zero. Corollary III.2.16 says that each such matrix is similar to one that is all zeroes except for blocks of subdiagonal ones.

2.3 Lemma If the matrices $T-\lambda I$ and N are similar then T and $N+\lambda I$ are also similar, via the same change of basis matrices.

Proof With $N = P(T-\lambda I)P^{-1} = PTP^{-1} - P(\lambda I)P^{-1}$ we have $N = PTP^{-1} - PP^{-1}(\lambda I)$ since the diagonal matrix λI commutes with anything, and so $N = PTP^{-1} - \lambda I$. Therefore $N + \lambda I = PTP^{-1}$. QED

2.4 Example The characteristic polynomial of

$$T = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}$$

is $(x-3)^2$ and so T has only the single eigenvalue 3. Thus for

$$T - 3I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$

the only eigenvalue is 0 and $T-3I$ is nilpotent. Finding the null spaces is routine; to ease this computation we take T to represent a transformation $t\colon \mathbb{C}^2 \to \mathbb{C}^2$ with respect to the standard basis (we shall do this for the rest of the chapter).

$$\mathscr{N}(t-3) = \{\begin{pmatrix} -y \\ y \end{pmatrix} \mid y \in \mathbb{C}\} \qquad \mathscr{N}((t-3)^2) = \mathbb{C}^2$$

The dimension of each null space shows that the action of the map $t-3$ on a string basis is $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$. Thus, here is the canonical form for $t-3$ with one choice for a string basis.

$$\mathrm{Rep}_{B,B}(t-3) = N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix} \rangle$$

By Lemma 2.3, T is similar to this matrix.

$$\mathrm{Rep}_{B,B}(t) = N + 3I = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

We can produce the similarity computation. Recall how to find the change of basis matrices P and P^{-1} to express N as $P(T-3I)P^{-1}$. The similarity diagram

$$\begin{array}{ccc} \mathbb{C}^2_{wrt\ \mathcal{E}_2} & \xrightarrow[T-3I]{t-3} & \mathbb{C}^2_{wrt\ \mathcal{E}_2} \\ {\scriptstyle id}\big\downarrow{\scriptstyle P} & & {\scriptstyle id}\big\downarrow{\scriptstyle P} \\ \mathbb{C}^2_{wrt\ B} & \xrightarrow[N]{t-3} & \mathbb{C}^2_{wrt\ B} \end{array}$$

describes that to move from the lower left to the upper left we multiply by

$$P^{-1} = \left(\mathrm{Rep}_{\mathcal{E}_2,B}(\mathrm{id})\right)^{-1} = \mathrm{Rep}_{B,\mathcal{E}_2}(\mathrm{id}) = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

and to move from the upper right to the lower right we multiply by this matrix.

$$P = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix}$$

So this equation expresses the similarity.

$$\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/4 & 1/4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

2.5 Example This matrix has characteristic polynomial $(x-4)^4$

$$T = \begin{pmatrix} 4 & 1 & 0 & -1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$$

and so has the single eigenvalue 4. The null space of $t-4$ has dimension two, the null space of $(t-4)^2$ has dimension three, and the null space of $(t-4)^3$ has dimension four. Thus, $t-4$ has the action on a string basis of $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{\beta}_3 \mapsto \vec{0}$ and $\vec{\beta}_4 \mapsto \vec{0}$. This gives the canonical form N for $t-4$, which in turn gives the form for t.

$$N + 4I = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

An array that is all zeroes, except for some number λ down the diagonal and blocks of subdiagonal ones, is a *Jordan block*. We have shown that Jordan block matrices are canonical representatives of the similarity classes of single-eigenvalue matrices.

2.6 Example The 3×3 matrices whose only eigenvalue is $1/2$ separate into three similarity classes. The three classes have these canonical representatives.

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \qquad \begin{pmatrix} 1/2 & 0 & 0 \\ 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

In particular, this matrix

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

belongs to the similarity class represented by the middle one, because we have adopted the convention of ordering the blocks of subdiagonal ones from the longest block to the shortest.

We will finish the program of this chapter by extending this work to cover maps and matrices with multiple eigenvalues. The best possibility for general maps and matrices would be if we could break them into a part involving their first eigenvalue λ_1 (which we represent using its Jordan block), a part with λ_2, etc.

This best possibility is what happens. For any transformation $t\colon V \to V$, we shall break the space V into the direct sum of a part on which $t-\lambda_1$ is nilpotent, a part on which $t-\lambda_2$ is nilpotent, etc.

Suppose that $t\colon V \to V$ is a linear transformation. The restriction of t to a subspace M need not be a linear transformation on M because there may be an $\vec{m} \in M$ with $t(\vec{m}) \notin M$ (for instance, the transformation that rotates the plane by a quarter turn does not map most members of the $x = y$ line subspace back within that subspace). To ensure that the restriction of a transformation to a part of a space is a transformation on the part we need the next condition.

2.7 Definition Let $t\colon V \to V$ be a transformation. A subspace M is t *invariant* if whenever $\vec{m} \in M$ then $t(\vec{m}) \in M$ (shorter: $t(M) \subseteq M$).

Recall that Lemma III.1.4 shows that for any transformation t on an n dimensional space the range spaces of iterates are stable

$$\mathscr{R}(t^n) = \mathscr{R}(t^{n+1}) = \cdots = \mathscr{R}_\infty(t)$$

as are the null spaces.

$$\mathscr{N}(t^n) = \mathscr{N}(t^{n+1}) = \cdots = \mathscr{N}_\infty(t)$$

Thus, the generalized null space $\mathscr{N}_\infty(t)$ and the generalized range space $\mathscr{R}_\infty(t)$ are t invariant. In particular, $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_i$ invariant.

The action of the transformation $t-\lambda_i$ on $\mathscr{N}_\infty(t-\lambda_i)$ is especially easy to understand. Observe that any transformation t is nilpotent on $\mathscr{N}_\infty(t)$, because if $\vec{v} \in \mathscr{N}_\infty(t)$ then by definition $t^n(\vec{v}) = \vec{0}$. Thus $t-\lambda_i$ is nilpotent on $\mathscr{N}_\infty(t-\lambda_i)$.

We shall take three steps to prove this section's major result. The next result is the first.

2.8 Lemma A subspace is t invariant if and only if it is $t-\lambda$ invariant for all scalars λ. In particular, if λ_i is an eigenvalue of a linear transformation t then for any other eigenvalue λ_j the spaces $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$ are $t-\lambda_j$ invariant.

Proof For the first sentence we check the two implications separately. The 'if' half is easy: if the subspace is $t-\lambda$ invariant for all scalars λ then using $\lambda = 0$ shows that it is t invariant. For 'only if' suppose that the subspace is t invariant, so that if $\vec{m} \in M$ then $t(\vec{m}) \in M$, and let λ be a scalar. The subspace M is closed under linear combinations and so if $t(\vec{m}) \in M$ then $t(\vec{m}) - \lambda\vec{m} \in M$. Thus if $\vec{m} \in M$ then $(t-\lambda)(\vec{m}) \in M$.

The lemma's second sentence follows from its first. The two spaces are $t-\lambda_i$ invariant so they are t invariant. Apply the first sentence again to conclude that they are also $t-\lambda_j$ invariant. QED

The second step of the three that we will take to prove this section's major result makes use of an additional property of $\mathscr{N}_\infty(t-\lambda_i)$ and $\mathscr{R}_\infty(t-\lambda_i)$, that they are complementary. Recall that if a space is the direct sum of two others $V = \mathscr{N} \oplus \mathscr{R}$ then any vector $\vec{v}$ in the space breaks into two parts $\vec{v} = \vec{n} + \vec{r}$ where $\vec{n} \in \mathscr{N}$ and $\vec{r} \in \mathscr{R}$, and recall also that if $B_{\mathscr{N}}$ and $B_{\mathscr{R}}$ are bases for $\mathscr{N}$ and $\mathscr{R}$ then the concatenation $B_{\mathscr{N}} {}^\frown B_{\mathscr{R}}$ is linearly independent. The next result says that for any subspaces $\mathscr{N}$ and $\mathscr{R}$ that are complementary as well as t invariant, the action of t on $\vec{v}$ breaks into the actions of t on $\vec{n}$ and on $\vec{r}$.

2.9 Lemma Let $t\colon V \to V$ be a transformation and let $\mathscr{N}$ and $\mathscr{R}$ be t invariant complementary subspaces of V. Then we can represent t by a matrix with blocks of square submatrices T_1 and T_2

$$\left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right) \begin{array}{l} \}\ \dim(\mathscr{N})\text{-many rows} \\ \}\ \dim(\mathscr{R})\text{-many rows} \end{array}$$

where Z_1 and Z_2 are blocks of zeroes.

Proof Since the two subspaces are complementary, the concatenation of a basis for $\mathscr{N}$ with a basis for $\mathscr{R}$ makes a basis $B = \langle \vec{v}_1, \dots, \vec{v}_p, \vec{\mu}_1, \dots, \vec{\mu}_q \rangle$ for V. We shall show that the matrix

$$\text{Rep}_{B,B}(t) = \begin{pmatrix} \vdots & & \vdots \\ \text{Rep}_B(t(\vec{v}_1)) & \cdots & \text{Rep}_B(t(\vec{\mu}_q)) \\ \vdots & & \vdots \end{pmatrix}$$

has the desired form.

Any vector $\vec{v} \in V$ is a member of $\mathscr{N}$ if and only if when it is represented with respect to B the final q coefficients are zero. As $\mathscr{N}$ is t invariant, each of the vectors $\text{Rep}_B(t(\vec{v}_1))$, ..., $\text{Rep}_B(t(\vec{v}_p))$ has this form. Hence the lower left of $\text{Rep}_{B,B}(t)$ is all zeroes. The argument for the upper right is similar. QED

To see that we have decomposed t into its action on the parts, let $B_{\mathscr{N}} = \langle \vec{v}_1, \dots, \vec{v}_p \rangle$ and $B_{\mathscr{R}} = \langle \vec{\mu}_1, \dots, \vec{\mu}_q \rangle$. The restrictions of t to the subspaces $\mathscr{N}$ and $\mathscr{R}$ are represented with respect to the bases $B_{\mathscr{N}}, B_{\mathscr{N}}$ and $B_{\mathscr{R}}, B_{\mathscr{R}}$ by the matrices T_1 and T_2. So with subspaces that are invariant and complementary we can split the problem of examining a linear transformation into two lower-dimensional subproblems. The next result illustrates this decomposition into blocks.

2.10 Lemma If T is a matrix with square submatrices T_1 and T_2

$$T = \left(\begin{array}{c|c} T_1 & Z_2 \\ \hline Z_1 & T_2 \end{array}\right)$$

where the Z's are blocks of zeroes, then $|T| = |T_1| \cdot |T_2|$.

Proof Suppose that T is $n \times n$, that T_1 is $p \times p$, and that T_2 is $q \times q$. In the permutation formula for the determinant

$$|T| = \sum_{\text{permutations } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$$

each term comes from a rearrangement of the column numbers $1, \ldots, n$ into a new order $\phi(1), \ldots, \phi(n)$. The upper right block Z_2 is all zeroes, so if a ϕ has at least one of $p+1, \ldots, n$ among its first p column numbers $\phi(1), \ldots, \phi(p)$ then the term arising from ϕ does not contribute to the sum because it is zero, e.g., if $\phi(1) = n$ then $t_{1,\phi(1)} t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0 \cdot t_{2,\phi(2)} \ldots t_{n,\phi(n)} = 0$.

So the above formula reduces to a sum over all permutations with two halves: any contributing ϕ is the composition of a ϕ_1 that rearranges only $1, \ldots, p$ and a ϕ_2 that rearranges only $p+1, \ldots, p+q$. Now, the distributive law and the fact that the signum of a composition is the product of the signums gives that this

$$|T_1| \cdot |T_2| = \Bigl(\sum_{\substack{\text{perms } \phi_1 \\ \text{of } 1,\ldots,p}} t_{1,\phi_1(1)} \cdots t_{p,\phi_1(p)} \operatorname{sgn}(\phi_1) \Bigr) \cdot \Bigl(\sum_{\substack{\text{perms } \phi_2 \\ \text{of } p+1,\ldots,p+q}} t_{p+1,\phi_2(p+1)} \cdots t_{p+q,\phi_2(p+q)} \operatorname{sgn}(\phi_2) \Bigr)$$

equals $|T| = \sum_{\text{contributing } \phi} t_{1,\phi(1)} t_{2,\phi(2)} \cdots t_{n,\phi(n)} \operatorname{sgn}(\phi)$. QED

2.11 Example

$$\begin{vmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} = 36$$

From Lemma 2.10 we conclude that if two subspaces are complementary and t invariant then t is one-to-one if and only if its restriction to each subspace is nonsingular.

Now for the promised third, and final, step to the main result.

2.12 Lemma If a linear transformation $t \colon V \to V$ has the characteristic polynomial $(x - \lambda_1)^{p_1} \ldots (x - \lambda_k)^{p_k}$ then (1) $V = \mathscr{N}_\infty(t - \lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t - \lambda_k)$ and (2) $\dim(\mathscr{N}_\infty(t - \lambda_i)) = p_i$.

Proof This argument consists of proving two preliminary claims, followed by proofs of clauses (1) and (2).

The first claim is that $\mathscr{N}_\infty(t - \lambda_i) \cap \mathscr{N}_\infty(t - \lambda_j) = \{\vec{0}\}$ when $i \neq j$. By Lemma 2.8 both $\mathscr{N}_\infty(t - \lambda_i)$ and $\mathscr{N}_\infty(t - \lambda_j)$ are t invariant. The intersection of t invariant subspaces is t invariant and so the restriction of t to $\mathscr{N}_\infty(t - \lambda_i) \cap \mathscr{N}_\infty(t - \lambda_j)$ is a linear transformation. Now, $t - \lambda_i$ is nilpotent on $\mathscr{N}_\infty(t - \lambda_i)$ and $t - \lambda_j$ is nilpotent on $\mathscr{N}_\infty(t - \lambda_j)$, so both $t - \lambda_i$ and $t - \lambda_j$ are nilpotent on the intersection. Therefore by Lemma 2.1 and the observation following it, if t has any eigenvalues on the intersection then the "only" eigenvalue is both λ_i and λ_j. This cannot be, so the restriction has no eigenvalues: $\mathscr{N}_\infty(t-\lambda_i)\cap\mathscr{N}_\infty(t-\lambda_j)$ is the trivial space (Lemma 3.10 shows that the only transformation that is without any eigenvalues is the transformation on the trivial space).

The second claim is that $\mathscr{N}_\infty(t-\lambda_i)\subseteq\mathscr{R}_\infty(t-\lambda_j)$, where $i\neq j$. To verify it we will show that $t-\lambda_j$ is one-to-one on $\mathscr{N}_\infty(t-\lambda_i)$ so that, since $\mathscr{N}_\infty(t-\lambda_i)$ is $t-\lambda_j$ invariant by Lemma 2.8, the map $t-\lambda_j$ is an automorphism of the subspace $\mathscr{N}_\infty(t-\lambda_i)$ and therefore that $\mathscr{N}_\infty(t-\lambda_i)$ is a subset of each $\mathscr{R}(t-\lambda_j)$, $\mathscr{R}((t-\lambda_j)^2)$, etc. For the verification that the map is one-to-one suppose that $\vec{v}\in\mathscr{N}_\infty(t-\lambda_i)$ is in the null space of $t-\lambda_j$, aiming to show that $\vec{v}=\vec{0}$. Consider the map $[(t-\lambda_i)-(t-\lambda_j)]^n$. On the one hand, the only vector that $(t-\lambda_i)-(t-\lambda_j)=\lambda_i-\lambda_j$ maps to zero is the zero vector. On the other hand, as in the proof of Lemma 1.7 we can apply the binomial expansion to get this.

$$(t-\lambda_i)^n(\vec{v})+\binom{n}{1}(t-\lambda_i)^{n-1}(t-\lambda_j)^1(\vec{v})+\binom{n}{2}(t-\lambda_i)^{n-2}(t-\lambda_j)^2(\vec{v})+\cdots$$

The first term is zero because $\vec{v}\in\mathscr{N}_\infty(t-\lambda_i)$ while the remaining terms are zero because $\vec{v}$ is in the null space of $t-\lambda_j$. Therefore $\vec{v}=\vec{0}$.

With those two preliminary claims done we can prove clause (1), that the space is the direct sum of the generalized null spaces. By Corollary III.2.2 the space is the direct sum $V=\mathscr{N}_\infty(t-\lambda_1)\oplus\mathscr{R}_\infty(t-\lambda_1)$. By the second claim $\mathscr{N}_\infty(t-\lambda_2)\subseteq\mathscr{R}_\infty(t-\lambda_1)$ and so we can get a basis for $\mathscr{R}_\infty(t-\lambda_1)$ by starting with a basis for $\mathscr{N}_\infty(t-\lambda_2)$ and adding extra basis elements taken from $\mathscr{R}_\infty(t-\lambda_1)\cap\mathscr{R}_\infty(t-\lambda_2)$. Thus $V=\mathscr{N}_\infty(t-\lambda_1)\oplus\mathscr{N}_\infty(t-\lambda_2)\oplus(\mathscr{R}_\infty(t-\lambda_1)\cap\mathscr{R}_\infty(t-\lambda_2))$. Continuing in this way we get this.

$$V=\mathscr{N}_\infty(t-\lambda_1)\oplus\cdots\oplus\mathscr{R}_\infty(t-\lambda_k)\oplus(\mathscr{R}_\infty(t-\lambda_1)\cap\cdots\cap\mathscr{R}_\infty(t-\lambda_k))$$

The first claim above shows that the final space is trivial.

We finish by verifying clause (2). Decompose V as $\mathscr{N}_\infty(t-\lambda_i)\oplus\mathscr{R}_\infty(t-\lambda_i)$ and apply Lemma 2.9.

$$T=\left(\begin{array}{c|c}T_1 & Z_2\\ \hline Z_1 & T_2\end{array}\right)\begin{array}{l}\}\ \dim(\,\mathscr{N}_\infty(t-\lambda_i)\,)\text{-many rows}\\ \}\ \dim(\,\mathscr{R}_\infty(t-\lambda_i)\,)\text{-many rows}\end{array}$$

Lemma 2.10 says that $|T-xI|=|T_1-xI|\cdot|T_2-xI|$. By the uniqueness clause of the Fundamental Theorem of Algebra, Theorem I.1.11, the determinants of the blocks have the same factors as the characteristic polynomial $|T_1-xI|=(x-\lambda_1)^{q_1}\cdots(x-\lambda_z)^{q_k}$ and $|T_2-xI|=(x-\lambda_1)^{r_1}\cdots(x-\lambda_z)^{r_k}$, where $q_1+r_1=p_1$, $\ldots$, $q_k+r_k=p_k$. We will finish by establishing that (i) $q_j=0$ for all $j\neq i$, and (ii) $q_i=p_i$. Together these prove clause (2) because they show that the degree of the polynomial $|T_1 - xI|$ is q_i and the degree of that polynomial equals the dimension of the generalized null space $\mathscr{N}_\infty(t-\lambda_i)$.

For (i), because the restriction of $t-\lambda_i$ to $\mathscr{N}_\infty(t-\lambda_i)$ is nilpotent on that space, t's only eigenvalue on that space is λ_i, by Lemma 2.2. So $q_j = 0$ for $j \neq i$.

For (ii), consider the restriction of t to $\mathscr{R}_\infty(t-\lambda_i)$. By Lemma III.2.1, the map $t-\lambda_i$ is one-to-one on $\mathscr{R}_\infty(t-\lambda_i)$ and so λ_i is not an eigenvalue of t on that subspace. Therefore $x-\lambda_i$ is not a factor of $|T_2 - xI|$, so $r_i = 0$, and so $q_i = p_i$. QED

Recall the goal of this chapter, to give a canonical form for matrix similarity. That result is next. It translates the above steps into matrix terms.

2.13 Theorem Any square matrix is similar to one in *Jordan form*

$$\begin{pmatrix} J_{\lambda_1} & & \textit{–zeroes–} & & \\ & J_{\lambda_2} & & & \\ & & \ddots & & \\ & & & J_{\lambda_{k-1}} & \\ & & \textit{–zeroes–} & & J_{\lambda_k} \end{pmatrix}$$

where each J_λ is the Jordan block associated with an eigenvalue λ of the original matrix (that is, each J_λ is all zeroes except for λ's down the diagonal and some subdiagonal ones).

Proof Given an $n\times n$ matrix T, consider the linear map $t\colon \mathbb{C}^n \to \mathbb{C}^n$ that it represents with respect to the standard bases. Use the prior lemma to write $\mathbb{C}^n = \mathscr{N}_\infty(t-\lambda_1) \oplus \cdots \oplus \mathscr{N}_\infty(t-\lambda_k)$ where $\lambda_1, \ldots, \lambda_k$ are the eigenvalues of t. Because each $\mathscr{N}_\infty(t-\lambda_i)$ is t invariant, Lemma 2.9 and the prior lemma show that t is represented by a matrix that is all zeroes except for square blocks along the diagonal. To make those blocks into Jordan blocks, pick each B_{λ_i} to be a string basis for the action of $t-\lambda_i$ on $\mathscr{N}_\infty(t-\lambda_i)$. QED

2.14 Corollary Every square matrix is similar to the sum of a diagonal matrix and a nilpotent matrix.

For Jordan form a canonical form for matrix similarity, strictly speaking it must be unique. That is, for any square matrix there needs to be one and only one matrix J similar to it and of the specified form. As stated the theorem allows us to rearrange the Jordan blocks. We could make this form unique, say by arranging the Jordan blocks so the eigenvalues are in order, and then arranging the blocks of subdiagonal ones from longest to shortest. Below, we won't bother with that.

2.15 Example This matrix has the characteristic polynomial $(x-2)^2(x-6)$.

$$T=\begin{pmatrix}2&0&1\\0&6&2\\0&0&2\end{pmatrix}$$

First we do the eigenvalue 2. Computation of the powers of $T-2I$, and of the null spaces and nullities, is routine. (Recall from Example 2.4 our convention of taking T to represent a transformation $t\colon\mathbb{C}^3\to\mathbb{C}^3$ with respect to the standard basis.)

p	$(T-2I)^p$	$\mathscr{N}((t-2)^p)$	*nullity*
1	$\begin{pmatrix}0&0&1\\0&4&2\\0&0&0\end{pmatrix}$	$\{\begin{pmatrix}x\\0\\0\end{pmatrix}\mid x\in\mathbb{C}\}$	1
2	$\begin{pmatrix}0&0&0\\0&16&8\\0&0&0\end{pmatrix}$	$\{\begin{pmatrix}x\\-z/2\\z\end{pmatrix}\mid x,z\in\mathbb{C}\}$	2
3	$\begin{pmatrix}0&0&0\\0&64&32\\0&0&0\end{pmatrix}$	*–same–*	*–same–*

So the generalized null space $\mathscr{N}_\infty(t-2)$ has dimension two. We know that the restriction of $t-2$ is nilpotent on this subspace. From the way that the nullities grow we know that the action of $t-2$ on a string basis is $\vec{\beta}_1\mapsto\vec{\beta}_2\mapsto\vec{0}$. Thus we can represent the restriction in the canonical form

$$N_2=\begin{pmatrix}0&0\\1&0\end{pmatrix}=\mathrm{Rep}_{B,B}(t-2)\qquad B_2=\langle\begin{pmatrix}1\\1\\-2\end{pmatrix},\begin{pmatrix}-2\\0\\0\end{pmatrix}\rangle$$

(other choices of basis are possible). Consequently, the action of the restriction of t to $\mathscr{N}_\infty(t-2)$ is represented by this matrix.

$$J_2=N_2+2I=\mathrm{Rep}_{B_2,B_2}(t)=\begin{pmatrix}2&0\\1&2\end{pmatrix}$$

The second eigenvalue is 6. Its computations are easier. Because the power of $x-6$ in the characteristic polynomial is one, the restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$

must be nilpotent, of index one (it can't be of index less than one and since $x-6$ is a factor of the characteristic polynomial with the exponent one it can't be of index more than one either). Its action on a string basis must be $\vec{\beta}_3 \mapsto \vec{0}$ and since it is the zero map, its canonical form N_6 is the 1×1 zero matrix. Consequently, the canonical form J_6 for the action of t on $\mathscr{N}_\infty(t-6)$ is the 1×1 matrix with the single entry 6. For the basis we can use any nonzero vector from the generalized null space.

$$B_6 = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle$$

Taken together, these two give that the Jordan form of T is

$$\mathrm{Rep}_{B,B}(t) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

where B is the concatenation of B_2 and B_6.

2.16 Example As a contrast with the prior example, this matrix

$$T = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

has the same characteristic polynomial $(x-2)^2(x-6)$, but here

p	$(T-6I)^p$	$\mathscr{N}((t-6)^p)$	*nullity*
1	$\begin{pmatrix} -4 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{pmatrix}$	$\{ \begin{pmatrix} x \\ (4/3)x \\ 0 \end{pmatrix} \mid x \in \mathbb{C} \}$	1
2	$\begin{pmatrix} 16 & -12 & -2 \\ 0 & 0 & -8 \\ 0 & 0 & 16 \end{pmatrix}$	*–same–*	—

the action of $t-2$ is stable after only one application—the restriction of $t-2$ to $\mathscr{N}_\infty(t-2)$ is nilpotent of index one. The restriction of $t-2$ to the generalized null space acts on a string basis via the two strings $\vec{\beta}_1 \mapsto \vec{0}$ and $\vec{\beta}_2 \mapsto \vec{0}$. We have this Jordan block associated with the eigenvalue 2.

$$J_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

So the contrast with the prior example is that while the characteristic polynomial tells us to look at the action of $t-2$ on its generalized null space, the characteristic polynomial does not completely describe $t-2$'s action. We must do some computations to find that the minimal polynomial is $(x-2)(x-6)$.

For the eigenvalue 6 the arguments for the second eigenvalue of the prior example apply again. The restriction of $t-6$ to $\mathscr{N}_\infty(t-6)$ is nilpotent of index one. Thus $t-6$'s canonical form N_6 is the 1×1 zero matrix, and the associated Jordan block J_6 is the 1×1 matrix with entry 6.

Therefore the Jordan form for T is a diagonal matrix.

$$\mathrm{Rep}_{B,B}(t)=\begin{pmatrix}2&0&0\\0&2&0\\0&0&6\end{pmatrix}\qquad B=B_2^\frown B_6=\langle\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\-2\end{pmatrix},\begin{pmatrix}2\\4\\0\end{pmatrix}\rangle$$

(Checking that the third vector in B is in the null space of $t-6$ is routine.)

2.17 Example A bit of computing with

$$T=\begin{pmatrix}-1&4&0&0&0\\0&3&0&0&0\\0&-4&-1&0&0\\3&-9&-4&2&-1\\1&5&4&1&4\end{pmatrix}$$

shows that its characteristic polynomial is $(x-3)^3(x+1)^2$. This table

p	$(T-3I)^p$	$\mathscr{N}((t-3)^p)$	*nullity*
1	$\begin{pmatrix}-4&4&0&0&0\\0&0&0&0&0\\0&-4&-4&0&0\\3&-9&-4&-1&-1\\1&5&4&1&1\end{pmatrix}$	$\{\begin{pmatrix}-(u+v)/2\\-(u+v)/2\\(u+v)/2\\u\\v\end{pmatrix}\mid u,v\in\mathbb{C}\}$	2
2	$\begin{pmatrix}16&-16&0&0&0\\0&0&0&0&0\\0&16&16&0&0\\-16&32&16&0&0\\0&-16&-16&0&0\end{pmatrix}$	$\{\begin{pmatrix}-z\\-z\\z\\u\\v\end{pmatrix}\mid z,u,v\in\mathbb{C}\}$	3
3	$\begin{pmatrix}-64&64&0&0&0\\0&0&0&0&0\\0&-64&-64&0&0\\64&-128&-64&0&0\\0&64&64&0&0\end{pmatrix}$	*–same–*	*–same–*

shows that the restriction of $t-3$ to $\mathscr{N}_\infty(t-3)$ acts on a string basis via the two strings $\vec{\beta}_1 \mapsto \vec{\beta}_2 \mapsto \vec{0}$ and $\vec{\beta}_3 \mapsto \vec{0}$.

A similar calculation for the other eigenvalue

p	$(T+1I)^p$	$\mathscr{N}((t+1)^p)$	*nullity*
1	$\begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 3 & -9 & -4 & 3 & -1 \\ 1 & 5 & 4 & 1 & 5 \end{pmatrix}$	$\{\begin{pmatrix} -(u+v) \\ 0 \\ -v \\ u \\ v \end{pmatrix} \mid u, v \in \mathbb{C}\}$	2
2	$\begin{pmatrix} 0 & 16 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 & 0 \\ 8 & -40 & -16 & 8 & -8 \\ 8 & 24 & 16 & 8 & 24 \end{pmatrix}$	*–same–*	*–same–*

gives that the restriction of $t+1$ to its generalized null space acts on a string basis via the two separate strings $\vec{\beta}_4 \mapsto \vec{0}$ and $\vec{\beta}_5 \mapsto \vec{0}$.

Therefore T is similar to this Jordan form matrix.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Exercises

2.18 Do the check for Example 2.4.

2.19 Each matrix is in Jordan form. State its characteristic polynomial and its minimal polynomial.

(a) $\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **(c)** $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ **(f)** $\begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & -4 \end{pmatrix}$ **(g)** $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(h) $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ **(i)** $\begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

✓ 2.20 Find the Jordan form from the given data.

(a) The matrix T is 5×5 with the single eigenvalue 3. The nullities of the powers are: $T-3I$ has nullity two, $(T-3I)^2$ has nullity three, $(T-3I)^3$ has nullity four, and $(T-3I)^4$ has nullity five.

(b) The matrix S is 5×5 with two eigenvalues. For the eigenvalue 2 the nullities are: $S-2I$ has nullity two, and $(S-2I)^2$ has nullity four. For the eigenvalue -1 the nullities are: $S+1I$ has nullity one.

2.21 Find the change of basis matrices for each example.

(a) Example 2.15 (b) Example 2.16 (c) Example 2.17

✓ 2.22 Find the Jordan form and a Jordan basis for each matrix.

(a) $\begin{pmatrix}-10&4\\-25&10\end{pmatrix}$ (b) $\begin{pmatrix}5&-4\\9&-7\end{pmatrix}$ (c) $\begin{pmatrix}4&0&0\\2&1&3\\5&0&4\end{pmatrix}$ (d) $\begin{pmatrix}5&4&3\\-1&0&-3\\1&-2&1\end{pmatrix}$

(e) $\begin{pmatrix}9&7&3\\-9&-7&-4\\4&4&4\end{pmatrix}$ (f) $\begin{pmatrix}2&2&-1\\-1&-1&1\\-1&-2&2\end{pmatrix}$ (g) $\begin{pmatrix}7&1&2&2\\1&4&-1&-1\\-2&1&5&-1\\1&1&2&8\end{pmatrix}$

✓ 2.23 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^2(x+2)^2$.

2.24 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-1)^3(x+2)$.

✓ 2.25 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^3(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

2.26 Find all possible Jordan forms of a transformation with characteristic polynomial $(x-2)^4(x+1)$ and minimal polynomial $(x-2)^2(x+1)$.

✓ 2.27 Diagonalize these.

(a) $\begin{pmatrix}1&1\\0&0\end{pmatrix}$ (b) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$

✓ 2.28 Find the Jordan matrix representing the differentiation operator on $\mathcal{P}_3$.

✓ 2.29 Decide if these two are similar.

$$\begin{pmatrix}1&-1\\4&-3\end{pmatrix}\qquad\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$$

2.30 Find the Jordan form of this matrix.

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

Also give a Jordan basis.

2.31 How many similarity classes are there for 3×3 matrices whose only eigenvalues are -3 and 4?

✓ 2.32 Prove that a matrix is diagonalizable if and only if its minimal polynomial has only linear factors.

2.33 Give an example of a linear transformation on a vector space that has no non-trivial invariant subspaces.

2.34 Show that a subspace is $t-\lambda_1$ invariant if and only if it is $t-\lambda_2$ invariant.

2.35 Prove or disprove: two $n \times n$ matrices are similar if and only if they have the same characteristic and minimal polynomials.

2.36 The *trace* of a square matrix is the sum of its diagonal entries.

(a) Find the formula for the characteristic polynomial of a 2×2 matrix.

(b) Show that trace is invariant under similarity, and so we can sensibly speak of the 'trace of a map'. (*Hint:* see the prior item.)

(c) Is trace invariant under matrix equivalence?

(d) Show that the trace of a map is the sum of its eigenvalues (counting multiplicities).

(e) Show that the trace of a nilpotent map is zero. Does the converse hold?

2.37 To use Definition 2.7 to check whether a subspace is t invariant, we seemingly have to check all of the infinitely many vectors in a (nontrivial) subspace to see if they satisfy the condition. Prove that a subspace is t invariant if and only if its subbasis has the property that for all of its elements, $t(\vec{\beta})$ is in the subspace.

✓ 2.38 Is t invariance preserved under intersection? Under union? Complementation? Sums of subspaces?

2.39 Give a way to order the Jordan blocks if some of the eigenvalues are complex numbers. That is, suggest a reasonable ordering for the complex numbers.

2.40 Let $\mathcal{P}_j(\mathbb{R})$ be the vector space over the reals of degree j polynomials. Show that if $j \leqslant k$ then $\mathcal{P}_j(\mathbb{R})$ is an invariant subspace of $\mathcal{P}_k(\mathbb{R})$ under the differentiation operator. In $\mathcal{P}_7(\mathbb{R})$, does any of $\mathcal{P}_0(\mathbb{R}), \ldots, \mathcal{P}_6(\mathbb{R})$ have an invariant complement?

2.41 In $\mathcal{P}_n(\mathbb{R})$, the vector space (over the reals) of degree n polynomials,

$$\mathcal{E} = \{p(x) \in \mathcal{P}_n(\mathbb{R}) \mid p(-x) = p(x) \text{ for all } x\}$$

and

$$\mathcal{O} = \{p(x) \in \mathcal{P}_n(\mathbb{R}) \mid p(-x) = -p(x) \text{ for all } x\}$$

are the *even* and the *odd* polynomials; $p(x) = x^2$ is even while $p(x) = x^3$ is odd. Show that they are subspaces. Are they complementary? Are they invariant under the differentiation transformation?

2.42 Lemma 2.9 says that if M and N are invariant complements then t has a representation in the given block form (with respect to the same ending as starting basis, of course). Does the implication reverse?

2.43 A matrix S is the *square root* of another T if $S^2 = T$. Show that any nonsingular matrix has a square root.

Topic

Method of Powers

In applications matrices can be large. Calculating eigenvalues and eigenvectors by finding and solving the characteristic polynomial is impractical, too slow and too error-prone. Some techniques avoid the characteristic polynomial. Here we shall see a method that is suitable for large matrices that are *sparse*, meaning that the great majority of the entries are zero.

Suppose that the $n \times n$ matrix T has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then $\mathbb{C}^n$ has a basis made of the associated eigenvectors $\langle \vec{\zeta}_1, \ldots, \vec{\zeta}_n \rangle$. For any $\vec{v} \in \mathbb{C}^n$, writing $\vec{v} = c_1\vec{\zeta}_1 + \cdots + c_n\vec{\zeta}_n$ and iterating T on $\vec{v}$ gives these.

$$\begin{aligned}
T\vec{v} &= c_1\lambda_1\vec{\zeta}_1 + c_2\lambda_2\vec{\zeta}_2 + \cdots + c_n\lambda_n\vec{\zeta}_n \\
T^2\vec{v} &= c_1\lambda_1^2\vec{\zeta}_1 + c_2\lambda_2^2\vec{\zeta}_2 + \cdots + c_n\lambda_n^2\vec{\zeta}_n \\
T^3\vec{v} &= c_1\lambda_1^3\vec{\zeta}_1 + c_2\lambda_2^3\vec{\zeta}_2 + \cdots + c_n\lambda_n^3\vec{\zeta}_n \\
&\vdots \\
T^k\vec{v} &= c_1\lambda_1^k\vec{\zeta}_1 + c_2\lambda_2^k\vec{\zeta}_2 + \cdots + c_n\lambda_n^k\vec{\zeta}_n
\end{aligned}$$

Assuming that $|\lambda_1|$ is the largest and dividing through

$$\frac{T^k\vec{v}}{\lambda_1^k} = c_1\vec{\zeta}_1 + c_2\frac{\lambda_2^k}{\lambda_1^k}\vec{\zeta}_2 + \cdots + c_n\frac{\lambda_n^k}{\lambda_1^k}\vec{\zeta}_n$$

shows that as k gets larger the fractions go to zero and so λ_1's term will dominate the expression and that expression has a limit of $c_1\vec{\zeta}_1$.

Thus if $c_1 \neq 0$, as k increases the vectors $T^k\vec{v}$ will tend toward the direction of the eigenvectors associated with the dominant eigenvalue. Consequently, the ratios of the vector lengths $|T^k\vec{v}|/|T^{k-1}\vec{v}|$ tend to that dominant eigenvalue.

For example, the eigenvalues of the matrix

$$T = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$$

are 3 and -1. If $\vec{v}$ has the components 1 and 1 then iterating gives this.

$\vec{v}$	$T\vec{v}$	$T^2\vec{v}$	$\cdots$	$T^9\vec{v}$	$T^{10}\vec{v}$
$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}3\\7\end{pmatrix}$	$\begin{pmatrix}9\\17\end{pmatrix}$	$\cdots$	$\begin{pmatrix}19\,683\\39\,367\end{pmatrix}$	$\begin{pmatrix}59\,049\\118\,097\end{pmatrix}$

The ratio between the lengths of the last two is $2.999\,9$.

We note two implementation issues. First, instead of finding the powers of T and applying them to $\vec{v}$, we will compute $\vec{v}_1$ as $T\vec{v}$ and then compute $\vec{v}_2$ as $T\vec{v}_1$, etc. (that is, we do not separately calculate T^2, T^3, ...). We can quickly do these matrix-vector products even if T is large, provided that it is sparse. The second issue is that to avoid generating numbers that are so large that they overflow our computer's capability, we can normalize the $\vec{v}_i$'s at each step. For instance, we can divide each $\vec{v}_i$ by its length (other possibilities are to divide it by its largest component, or simply by its first component). We thus implement this method by generating

$$\begin{aligned}
\vec{w}_0 &= \vec{v}_0/|\vec{v}_0| \\
\vec{v}_1 &= T\vec{w}_0 \\
\vec{w}_1 &= \vec{v}_1/|\vec{v}_1| \\
\vec{v}_2 &= T\vec{w}_2 \\
&\vdots \\
\vec{w}_{k-1} &= \vec{v}_{k-1}/|\vec{v}_{k-1}| \\
\vec{v}_k &= T\vec{w}_k
\end{aligned}$$

until we are satisfied. Then $\vec{v}_k$ is an approximation of an eigenvector, and the approximation of the dominant eigenvalue is the ratio $(T \bullet \vec{v}_k)/(\vec{v}_k \bullet \vec{v}_k) \approx (\lambda_1\vec{v}_k \cdot \vec{v}_k)/(\vec{v}_k \bullet \vec{v}_k) = \lambda_1$.

One way that we could be 'satisfied' is to iterate until our approximation of the eigenvalue settles down. We could decide for instance to stop the iteration process not after some fixed number of steps, but instead when $|\vec{v}_k|$ differs from $|\vec{v}_{k-1}|$ by less than one percent, or when they agree up to the second significant digit.

The rate of convergence is determined by the rate at which the powers of $|\lambda_2/\lambda_1|$ go to zero, where λ_2 is the eigenvalue of second largest length. If that ratio is much less than one then convergence is fast but if it is only slightly less than one then convergence can be quite slow. Consequently, the method of powers is not the most commonly used way of finding eigenvalues (although it is the simplest one, which is why it is here). Instead, there are a variety of methods that generally work by first replacing the given matrix T with another that is similar to it and so has the same eigenvalues, but is in some reduced form such as tridiagonal form, where the only nonzero entries are on the diagonal, or just above or below it. Then special case techniques can find the eigenvalues. Once we know the eigenvalues then we can easily compute the eigenvectors of T. These other methods are outside of our scope. A good reference is [Goult, *et al.*]

Exercises

1 Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1 and 2. Compare the answer with the one obtained by solving the characteristic equation.

(a) $\begin{pmatrix} 1 & 5 \\ 0 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

2 Redo the prior exercise by iterating until $|\vec{v}_k| - |\vec{v}_{k-1}|$ has absolute value less than 0.01 At each step, normalize by dividing each vector by its length. How many iterations does it take? Are the answers significantly different?

3 Use ten iterations to estimate the largest eigenvalue of these matrices, starting from the vector with components 1, 2, and 3. Compare the answer with the one obtained by solving the characteristic equation.

(a) $\begin{pmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}$

4 Redo the prior exercise by iterating until $|\vec{v}_k| - |\vec{v}_{k-1}|$ has absolute value less than 0.01. At each step, normalize by dividing each vector by its length. How many iterations does it take? Are the answers significantly different?

5 What happens if $c_1 = 0$? That is, what happens if the initial vector does not to have any component in the direction of the relevant eigenvector?

6 How can we adapt the method of powers to find the smallest eigenvalue?

Computer Code

This is the code for the computer algebra system Octave that did the calculation above. (It has been lightly edited to remove blank lines, etc.)

```
>T=[3, 0;
    8, -1]
T=
   3   0
   8  -1
>v0=[1; 2]
v0=
   1
   1
>v1=T*v0
v1=
   3
   7
>v2=T*v1
v2=
   9
  17
>T9=T**9
T9=
  19683  0
```

```
  39368 -1
>T10=T**10
T10=
  59049  0
 118096  1
>v9=T9*v0
v9=
  19683
  39367
>v10=T10*v0
v10=
  59049
 118096
>norm(v10)/norm(v9)
ans=2.9999
```

Remark. This does not use the full power of Octave; it has built-in functions to automatically apply sophisticated methods to find eigenvalues and eigenvectors.

Topic

Stable Populations

Imagine a reserve park with animals from a species that we are protecting. The park doesn't have a fence so animals cross the boundary, both from the inside out and from the outside in. Every year, 10% of the animals from inside of the park leave and 1% of the animals from the outside find their way in. Can we reach a stable level; are there populations for the park and the rest of the world that will stay constant over time, with the number of animals leaving equal to the number of animals entering?

Let p_n be the year n population in the park and let r_n be the population in the rest of the world.

$$\begin{aligned} p_{n+1} &= .90p_n + .01r_n \\ r_{n+1} &= .10p_n + .99r_n \end{aligned}$$

We have this matrix equation.

$$\begin{pmatrix} p_{n+1} \\ r_{n+1} \end{pmatrix} = \begin{pmatrix} .90 & .01 \\ .10 & .99 \end{pmatrix} \begin{pmatrix} p_n \\ r_n \end{pmatrix}$$

The population will be stable if $p_{n+1} = p_n$ and $r_{n+1} = r_n$ so that the matrix equation $\vec{v}_{n+1} = T\vec{v}_n$ becomes $\vec{v} = T\vec{v}$. We are therefore looking for eigenvectors for T that are associated with the eigenvalue $\lambda = 1$. The equation $\vec{0} = (\lambda I - T)\vec{v} = (I - T)\vec{v}$ is

$$\begin{pmatrix} 0.10 & -0.01 \\ -0.10 & 0.01 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and gives the eigenspace of vectors with the restriction that $p = .1r$. For example, if we start with a park population $p = 10\,000$ animals and a rest of the world population of $r = 100\,000$ animals then every year ten percent of those inside leave the park (this is a thousand animals), and every year one percent of those from the rest of the world enter the park (also a thousand animals). The population is stable, self-sustaining.

Now imagine that we are trying to raise the total world population of this species. We are trying to have the world population grow at 1% per year. This makes the population level stable in some sense, although it is a dynamic stability, in contrast to the static population level of the $\lambda = 1$ case. The equation $\vec{v}_{n+1} = 1.01 \cdot \vec{v}_n = T\vec{v}_n$ leads to $((1.01I - T)\vec{v} = \vec{0}$, which gives this system.

$$\begin{pmatrix} 0.11 & -0.01 \\ -0.10 & 0.02 \end{pmatrix} \begin{pmatrix} p \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This matrix is nonsingular and so the only solution is $p = 0$, $r = 0$. Thus there is no nontrivial initial population that would lead to a regular annual one percent growth rate in p and r.

We can look for the rates that allow an initial population for the park that results in a steady growth behavior. We consider $\lambda\vec{v} = T\vec{v}$ and solve for λ.

$$0 = \begin{vmatrix} \lambda - .9 & .01 \\ .10 & \lambda - .99 \end{vmatrix} = (\lambda - .9)(\lambda - .99) - (.10)(.01) = \lambda^2 - 1.89\lambda + .89$$

We already know that $\lambda = 1$ is one solution of this characteristic equation. The other is 0.89. Thus there are two ways to have a dynamically stable p and r, where the two grow at the same rate despite the leaky park boundaries: have a world population that is does not grow or shrink, and have a world population that shrinks by 11% every year.

So one way to look at eigenvalues and eigenvectors is that they give a stable state for a system. If the eigenvalue is one then the system is static and if the eigenvalue isn't one then it is a dynamic stability.

Exercises

1 For the park discussed above, what should be the initial park population in the case where the populations decline by 11% every year?

2 What will happen to the population of the park in the event of a growth in world population of 1% per year? Will it lag the world growth, or lead it? Assume that the initial park population is ten thousand, and the world population is one hundred thousand, and calculate over a ten year span.

3 The park discussed above is partially fenced so that now, every year, only 5% of the animals from inside of the park leave (still, about 1% of the animals from the outside find their way in). Under what conditions can the park maintain a stable population now?

4 Suppose that a species of bird only lives in Canada, the United States, or in Mexico. Every year, 4% of the Canadian birds travel to the US, and 1% of them travel to Mexico. Every year, 6% of the US birds travel to Canada, and 4% go to Mexico. From Mexico, every year 10% travel to the US, and 0% go to Canada.

(a) Give the transition matrix.

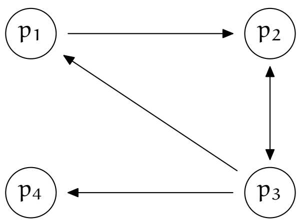
(b) Is there a way for the three countries to have constant populations?

Topic

Page Ranking

Imagine that you are looking for the best book on Linear Algebra. You probably would try a web search engine such as Google. These lists pages ranked by importance. The ranking is defined, as Google's founders have said in [Brin & Page], that a page is important if other important pages link to it: "a page can have a high PageRank if there are many pages that point to it, or if there are some pages that point to it and have a high PageRank." But isn't that circular—how can they tell whether a page is important without first deciding on the important pages? With eigenvalues and eigenvectors.

We will present a simplified version of the Page Rank algorithm. For that we will model the World Wide Web as a collection of pages connected by links. This diagram, from [Wills], shows the pages as circles, and the links as arrows; for instance, page p_1 has a link to page p_2.



The key idea is that pages that should be highly ranked if they are cited often by other pages. That is, we raise the importance of a page p_i if it is linked-to from page p_j. The increment depends on the importance of the linking page p_j divided by how many out-links a_j are on that page.

$$\mathcal{I}(p_i) = \sum_{\text{in-linking pages } p_j} \frac{\mathcal{I}(p_j)}{a_j}$$

This matrix stores the information.

$$\begin{pmatrix} 0 & 0 & 1/3 & 0 \\ 1 & 0 & 1/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{pmatrix}$$

The algorithm's inventors describe a way to think about that matrix.

> PageRank can be thought of as a model of user behavior. We assume there is a 'random surfer' who is given a web page at random and keeps clicking on links, never hitting "back" ... The probability that the random surfer visits a page is its PageRank. [Brin & Page]

In the diagram, a surfer on page p_3 has a probability $1/3$ of going next to each of the other pages.

That leads us to the problem of page p_4. Many pages are *dangling* or *sink links*, without any outbound links. The simplest model of what happens here is to imagine that the surfer goes to a next page entirely at random.

$$H = \begin{pmatrix} 0 & 0 & 1/3 & 1/4 \\ 1 & 0 & 1/3 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \end{pmatrix}$$

We will find vector $\vec{\mathcal{I}}$ whose components are the importance rankings of each page $\mathcal{I}(p_i)$. With this notation, our requirements for the page rank are that $H\vec{\mathcal{I}} = \vec{\mathcal{I}}$. That is, we want an eigenvector of the matrix associated with the eigenvalue $\lambda = 1$.

Here is *Sage*'s calculation of the eigenvectors (slightly edited to fit on the page).

```
sage: H=matrix([[0,0,1/3,1/4], [1,0,1/3,1/4], [0,1,0,1/4], [0,0,1/3,1/4]])
sage: H.eigenvectors_right()
[(1, [
(1, 2, 9/4, 1)
], 1), (0, [
(0, 1, 3, -4)
], 1), (-0.3750000000000000? - 0.4389855730355308?*I,
         [(1, -0.1250000000000000? + 1.316956719106593?*I,
          -1.875000000000000? - 1.316956719106593?*I, 1)], 1),
       (-0.3750000000000000? + 0.4389855730355308?*I,
         [(1, -0.1250000000000000? - 1.316956719106593?*I,
           -1.875000000000000? + 1.316956719106593?*I, 1)], 1)]
```

The eigenvector that *Sage* gives associated with the eigenvalue $\lambda = 1$ is this.

$$\begin{pmatrix} 1 \\ 2 \\ 9/4 \\ 1 \end{pmatrix}$$

Of course, there are many vectors in that eigenspace. To get a page rank number we normalize to length one.

```
sage: v=vector([1, 2, 9/4, 1])
sage: v/v.norm()
(4/177*sqrt(177), 8/177*sqrt(177), 3/59*sqrt(177), 4/177*sqrt(177))
sage: w=v/v.norm()
sage: w.n()
(0.300658411201132, 0.601316822402263, 0.676481425202546, 0.300658411201132)
```

So we rank the first and fourth pages as of equal importance. We rank the second and third pages as much more important than those, and about equal in importance as each other.

We'll add one more refinement. We will allow the surfer to pick a new page at random even if they are not on a dangling page. Let this happen with probability α.

$$G = \alpha \cdot \begin{pmatrix} 0 & 0 & 1/3 & 1/4 \\ 1 & 0 & 1/3 & 1/4 \\ 0 & 1 & 0 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \end{pmatrix} + (1-\alpha) \cdot \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

This is the *Google matrix*.

In practice α is typically between 0.85 and 0.99. Here are the ranks for the four pages with various α's.

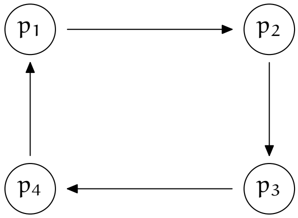
α	0.85	0.90	0.95	0.99
p_1	0.325	0.317	0.309	0.302
p_2	0.602	0.602	0.602	0.601
p_3	0.652	0.661	0.669	0.675
p_4	0.325	0.317	0.309	0.302

The details of the algorithms used by commercial search engines are secret, no doubt have many refinements, and also change frequently. But the inventors of Google were gracious enough to outline the basis for their work in [Brin & Page]. A more current source is [Wikipedia, Google Page Rank]. Two additional excellent expositions are [Wills] and [Austin].

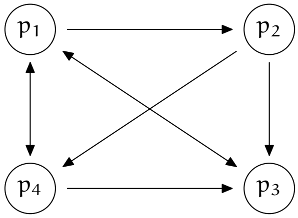
Exercises

1 A square matrix is *stochastic* if the sum of the entries in each column is one. The Google matrix is computed by taking a combination $G = \alpha * H + (1-\alpha) * S$ of two stochastic matrices. Show that G must be stochastic.

2 For this web of pages, the importance of each page should be equal. Verify it for $\alpha = 0.85$.



3 [Bryan & Leise] Give the importance ranking for this web of pages.



(a) Use $\alpha = 0.85$.

(b) Use $\alpha = 0.95$.

(c) Observe that while p_3 is linked-to from all other pages, and therefore seems important, it is not the highest ranked page. What is the highest ranked page? Explain.

Topic

Linear Recurrences

In 1202 Leonardo of Pisa, known as Fibonacci, posed this problem.

> A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?

This moves past an elementary exponential growth model for populations to include that newborns are not fertile for some period, here a month. However, it retains other simplifying assumptions such as that there is an age after which the rabbits are infertile.

To get next month's total number of pairs we add the number of pairs alive going into next month to the number of pairs that will be newly born next month. The latter equals the number of pairs that will be productive going into next month, which is the number that next month will have been alive for at least two months.

$$F(n) = F(n-1) + F(n-2) \qquad \text{where } F(0) = 0,\ F(1) = 1 \tag{$*$}$$

On the left is a *recurrence relation*. It gets that name because F recurs in its own defining equation. On the right are the initial conditions. From $(*)$ we can compute $F(2)$, $F(3)$, etc., to work up to the answer for Fibonacci's question.

month n	0	1	2	3	4	5	6	7	8	9	10	11	12
pairs $F(n)$	0	1	1	2	3	5	8	13	21	34	55	89	144

We will use linear algebra to get a formula that calculates $F(n)$ without having to first calculate the intermediate values $F(2)$, $F(3)$, etc.

We start by giving $(*)$ a matrix formulation.

$$\begin{pmatrix} F(n) \\ F(n-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F(n-1) \\ F(n-2) \end{pmatrix} \qquad \text{where } \begin{pmatrix} F(1) \\ F(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Write T for the matrix and $\vec{v}_n$ for the vector with components $F(n)$ and $F(n-1)$ so that $\vec{v}_n = T^{n-1}\vec{v}_1$ for $n \geqslant 1$. If we diagonalize T then we have a fast way to compute its powers: where $T = PDP^{-1}$ then $T^n = PD^nP^{-1}$ and the n-th power of the diagonal matrix D is the diagonal matrix whose entries are the n-th powers of the entries of D.

The characteristic equation of T is $\lambda^2 - \lambda - 1 = 0$. The quadratic formula gives its roots as $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. (These are sometimes called "golden ratios;" see [Falbo].) Diagonalizing gives this.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -(\frac{1-\sqrt{5}}{2\sqrt{5}}) \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$$

Introducing the vectors and taking the n-th power, we have

$$\begin{aligned} \begin{pmatrix} F(n) \\ F(n-1) \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} f(1) \\ f(0) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -(\frac{1-\sqrt{5}}{2\sqrt{5}}) \\ \frac{-1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

The calculation is ugly but not hard.

$$\begin{aligned} \begin{pmatrix} F(n) \\ F(n-1) \end{pmatrix} &= \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\ -\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{pmatrix} \end{aligned}$$

We want the first component.

$$F(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]$$

This formula gives the value of any member of the sequence without having to first find the intermediate values.

Because $(1-\sqrt{5})/2 \approx -0.618$ has absolute value less than one, its powers go to zero and so the $F(n)$ formula is dominated by its first term. Although we have extended the elementary model of population growth by adding a delay period before the onset of fertility, we nonetheless still get a function that is asymptotically exponential.

In general, a *homogeneous linear recurrence relation of order* k has this form.

$$f(n) = a_{n-1}f(n-1) + a_{n-2}f(n-2) + \cdots + a_{n-k}f(n-k)$$

This recurrence relation is homogeneous because it has no constant term, i.e, we can rewrite it as $0 = -f(n) + a_{n-1}f(n-1) + a_{n-2}f(n-2) + \cdots + a_{n-k}f(n-k)$. It is of order k because it uses k-many prior terms to calculate $f(n)$. The relation, cimbined with initial conditions giving values for $f(0)$, ..., $f(k-1)$, completely determines a sequence, simply because we can compute $f(n)$ by first computing $f(k)$, $f(k+1)$, etc. As with the Fibonacci case we will find a formula that solves the recurrence, that directly gives $f(n)$

Let V be the set of functions with domain $\mathbb{N} = \{0, 1, 2, \ldots\}$ and codomain $\mathbb{C}$. (Where convenient we sometimes use the domain $\mathbb{Z}^+ = \{1, 2, \ldots\}$.) This is a vector space under the usual meaning for addition and scalar multiplication, that $f + g$ is the map $x \mapsto f(x) + g(x)$ and cf is the map $x \mapsto c \cdot f(x)$.

If we put aside any initial conditions and look only at the recurrence, then there may be many functions satisfying the relation. For example, the Fibonacci recurrence that each value beyond the initial ones is the sum of the prior two is satisfied by the function L whose first few values are $L(0) = 2$, $L(1) = 1$, $L(2) = 3$, $L(3) = 4$, and $L(4) = 7$.

Fix a homogeneous linear recurrence relation of order k and consider the subset S of functions satisfying the relation (without initial conditions). This S is a subspace of V. It is nonempty because the zero function is a solution, by homogeneity. It is closed under addition because if f_1 and f_2 are solutions then this holds.

$$\begin{aligned}
-(f_1+f_2)(n) + a_{n-1}(f_1+f_2)(n-1) + \cdots + a_{n-k}(f_1+f_2)(n-k)& \\
&= (-f_1(n) + \cdots + a_{n-k}f_1(n-k)) \\
&\quad\; + (-f_2(n) + \cdots + a_{n-k}f_2(n-k)) \\
&= 0 + 0 = 0
\end{aligned}$$

It is also closed under scalar multiplication.

$$\begin{aligned}
-(rf_1)(n) + a_{n-1}(rf_1)(n-1) + \cdots + a_{n-k}(rf_1)(n-k)& \\
&= r \cdot (-f_1(n) + \cdots + a_{n-k}f_1(n-k)) \\
&= r \cdot 0 \\
&= 0
\end{aligned}$$

We can find the dimension of S. Where k is the order of the recurrence, consider this map from the set of functions S to the set of k-tall vectors.

$$f \mapsto \begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(k-1) \end{pmatrix}$$

Exercise 4 shows that this is linear. Any solution of the recurrence is uniquely determined by the k-many initial conditions so this map is one-to-one and onto. Thus it is an isomorphism, and S has dimension k.

So we can describe the set of solutions of our linear homogeneous recurrence relation of order k by finding a basis consisting of k-many linearly independent functions. To produce those we give the recurrence a matrix formulation.

$$\begin{pmatrix} f(n) \\ f(n-1) \\ \vdots \\ f(n-k+1) \end{pmatrix} = \begin{pmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \ldots & a_{n-k+1} & a_{n-k} \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \begin{pmatrix} f(n-1) \\ f(n-2) \\ \vdots \\ f(n-k) \end{pmatrix}$$

Call the matrix A. We want its characteristic function, the determinant of $A - \lambda I$. The pattern in the 2×2 case

$$\begin{pmatrix} a_{n-1}-\lambda & a_{n-2} \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - a_{n-1}\lambda - a_{n-2}$$

and the 3×3 case

$$\begin{pmatrix} a_{n-1}-\lambda & a_{n-2} & a_{n-3} \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{pmatrix} = -\lambda^3 + a_{n-1}\lambda^2 + a_{n-2}\lambda + a_{n-3}$$

leads us to expect, and Exercise 5 verifies, that this is the characteristic equation.

$$\begin{aligned} 0 &= \begin{vmatrix} a_{n-1}-\lambda & a_{n-2} & a_{n-3} & \ldots & a_{n-k+1} & a_{n-k} \\ 1 & -\lambda & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\lambda & & & \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -\lambda \end{vmatrix} \\ &= \pm(-\lambda^k + a_{n-1}\lambda^{k-1} + a_{n-2}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k}) \end{aligned}$$

The $\pm$ is not relevant to find the roots so we drop it. We say that the polynomial $-\lambda^k + a_{n-1}\lambda^{k-1} + a_{n-2}\lambda^{k-2} + \cdots + a_{n-k+1}\lambda + a_{n-k}$ is associated with the recurrence relation.

If the characteristic equation has no repeated roots then the matrix is diagonalizable and we can, in theory, get a formula for $f(n)$, as in the Fibonacci case. But because we know that the subspace of solutions has dimension k we do not need to do the diagonalization calculation, provided that we can exhibit k different linearly independent functions satisfying the relation.

Where r_1, r_2, ..., r_k are the distinct roots, consider the functions of powers of those roots, $f_{r_1}(n) = r_1^n$ through $f_{r_k}(n) = r_k^n$. Exercise 6 shows that each is a solution of the recurrence and that they form a linearly independent set. So, if the roots of the associated polynomial are distinct, any solution of the relation has the form $f(n) = c_1r_1^n + c_2r_2^n + \cdots + c_kr_k^n$ for some scalars $c_1, \ldots, c_n$. (The case of repeated roots is similar but we won't cover it here; see any text on Discrete Mathematics.)

Now we bring in the initial conditions. Use them to solve for c_1, ..., c_n. For instance, the polynomial associated with the Fibonacci relation is $-\lambda^2 + \lambda + 1$, whose roots are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$ and so any solution of the Fibonacci recurrence has the form $f(n) = c_1((1 + \sqrt{5})/2)^n + c_2((1 - \sqrt{5})/2)^n$. Use the Fibonacci initial conditions for $n = 0$ and $n = 1$

$$\begin{aligned} c_1 + \phantom{(1-\sqrt{5}/2)}c_2 &= 0 \\ (1 + \sqrt{5}/2)c_1 + (1 - \sqrt{5}/2)c_2 &= 1 \end{aligned}$$

and solve to get $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, as we found above.

We close by considering the nonhomogeneous case, where the relation has the form $f(n + 1) = a_nf(n) + a_{n-1}f(n - 1) + \cdots + a_{n-k}f(n - k) + b$ for some nonzero b. We only need a small adjustment to make the transition from the homogeneous case.

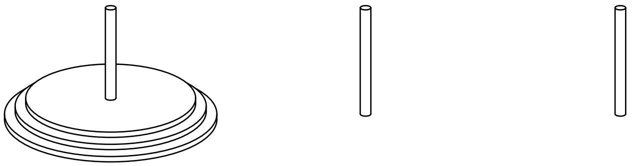
This classic example illustrates: in 1883, Edouard Lucas posed the Tower of Hanoi problem.

> In the great temple at Benares, beneath the dome which marks the center of the world, rests a brass plate in which are fixed three diamond needles, each a cubit high and as thick as the body of a bee. On one of these needles, at the creation, God placed sixty four disks of pure gold, the largest disk resting on the brass plate, and the others getting smaller and smaller up to the top one. This is the Tower of Brahma. Day and night unceasingly the priests transfer the disks from one diamond needle to another according to the fixed and immutable laws of Bram-ah, which require that the priest on duty must not move more than one disk at a time and that he must

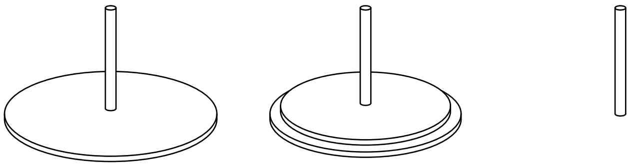
> place this disk on a needle so that there is no smaller disk below it. When the sixty-four disks shall have been thus transferred from the needle on which at the creation God placed them to one of the other needles, tower, temple, and Brahmins alike will crumble into dusk, and with a thunderclap the world will vanish. (Translation of [De Parville] from [Ball & Coxeter].)

We put aside the question of why the priests don't sit down for a while and have the world last a little longer, and instead ask how many disk moves it will take. Before tackling the sixty four disk problem we will consider the problem for three disks.

To begin, all three disks are on the same needle.



After the three moves of taking the small disk to the far needle, the mid-sized disk to the middle needle, and then the small disk to the middle needle, we have this.



Now we can move the big disk to the far needle. Then to finish we repeat the three-move process on the two smaller disks, this time so that they end up on the third needle, on top of the big disk.

That sequence of moves is the best that we can do. To move the bottom disk at a minimum we must first move the smaller disks to the middle needle, then move the big one, and then move all the smaller ones from the middle needle to the ending needle. Since this minimum suffices, we get this recurrence.

$$T(n) = T(n-1) + 1 + T(n-1) = 2T(n-1) + 1 \qquad \text{where } T(1) = 1$$

Here are the first few values of T.

disks n	1	2	3	4	5	6	7	8	9	10
moves $T(n)$	1	3	7	15	31	63	127	255	511	1023

Of course, these numbers are one less than a power of two. To derive this write the original relation as $-1 = -T(n)+2T(n-1)$. Consider $0 = -T(n)+2T(n-1)$, a linear homogeneous recurrence of order 1. Its associated polynomial is $-\lambda+2$, with the single root $r_1=2$. Thus functions satisfying the homogeneous relation take the form $c_1 2^n$.

That's the homogeneous solution. Now we need a particular solution. Because the nonhomogeneous relation $-1=-T(n)+2T(n-1)$ is so simple, we can by eye spot a particular solution $T(n)=-1$. Any solution of the recurrence $T(n)=2T(n-1)+1$ (without initial conditions) is the sum of the homogeneous solution and the particular solution: $c_1 2^n-1$. Now the initial condition $T(1)=1$ gives that $c_1=1$ and we've gotten the formula that generates the table: the n-disk Tower of Hanoi problem requires $T(n)=2^n-1$ moves.

Finding a particular solution in more complicated cases is, perhaps not surprisingly, more complicated. A delightful and rewarding, but challenging, source is [Graham, Knuth, Patashnik]. For more on the Tower of Hanoi see [Ball & Coxeter], [Gardner 1957], and [Hofstadter]. Some computer code follows the exercises.

Exercises

1 How many months until the number of Fibonacci rabbit pairs passes a thousand? Ten thousand? A million?

2 Solve each homogeneous linear recurrence relations.
 (a) $f(n)=5f(n-1)-6f(n-2)$
 (b) $f(n)=4f(n-2)$
 (c) $f(n)=5f(n-1)-2f(n-2)-8f(n-3)$

3 Give a formula for the relations of the prior exercise, with these initial conditions.
 (a) $f(0)=1$, $f(1)=1$
 (b) $f(0)=0$, $f(1)=1$
 (c) $f(0)=1$, $f(1)=1$, $f(2)=3$.

4 Check that the isomorphism given between S and $\mathbb{R}^k$ is a linear map.

5 Show that the characteristic equation of the matrix is as stated, that is, is the polynomial associated with the relation. (*Hint:* expanding down the final column and using induction will work.)

6 Given a homogeneous linear recurrence relation $f(n)=a_n f(n-1)+\cdots+a_{n-k}f(n-k)$, let $r_1,\ldots,r_k$ be the roots of the associated polynomial. Prove that each function $f_{r_i}(n)=r_k^n$ satisfies the recurrence (without initial conditions).

7 (This refers to the value $T(64)=18,446,744,073,709,551,615$ given in the computer code below.) Transferring one disk per second, how many years would it take the priests at the Tower of Hanoi to finish the job?

Computer Code

This code generates the first few values of a function defined by a recurrence and initial conditions. It is in the Scheme dialect of LISP, specifically, [Chicken Scheme].

After loading an extension that keeps the computer from switching to floating point numbers when the integers get large, the Tower of Hanoi function is straightforward.

```
(require-extension numbers)

(define (tower-of-hanoi-moves n)
    (if (= n 1)
       1
       (+ (* (tower-of-hanoi-moves (- n 1))
             2)
          1) ) )

; Two helper funcitons
(define (first-few-outputs proc n)
    (first-few-outputs-aux proc n '()) )

(define (first-few-outputs-aux proc n lst)
    (if (< n 1)
    lst
    (first-few-outputs-aux proc (- n 1) (cons (proc n) lst)) ) )
```

(For readers unused to recursive code: to compute $\mathrm{T}(64)$, the computer wants to compute $2 * \mathrm{T}(63) - 1$, which requires computing $\mathrm{T}(63)$. The computer puts the 'times 2' and the 'plus 1' aside for a moment. It computes $\mathrm{T}(63)$ by using this same piece of code (that's what 'recursive' means), and to do that it wants to compute $2 * \mathrm{T}(62) - 1$. This keeps up until, after 63 steps, the computer tries to compute $\mathrm{T}(1)$. It then returns $\mathrm{T}(1) = 1$, which allows the computation of $\mathrm{T}(2)$ to proceed, etc., until the original computation of $\mathrm{T}(64)$ finishes.)

The helper functions give a table of the first few values. Here is the session at the prompt.

```
#;1> (load "hanoi.scm")
; loading hanoi.scm ...
; loading /var/lib//chicken/6/numbers.import.so ...
; loading /var/lib//chicken/6/chicken.import.so ...
; loading /var/lib//chicken/6/foreign.import.so ...
; loading /var/lib//chicken/6/numbers.so ...
#;2> (tower-of-hanoi-moves 64)
18446744073709551615
#;3> (first-few-outputs tower-of-hanoi-moves 64)
(1 3 7 15 31 63 127 255 511 1023 2047 4095 8191 16383 32767 65535 131071 262143 524287 1048575
2097151 4194303 8388607 16777215 33554431 67108863 134217727 268435455 536870911 1073741823
2147483647 4294967295 8589934591 17179869183 34359738367 68719476735 137438953471 274877906943
549755813887 1099511627775 2199023255551 4398046511103 8796093022207 17592186044415
35184372088831 70368744177663 140737488355327 281474976710655 562949953421311 1125899906842623
2251799813685247 4503599627370495 9007199254740991 18014398509481983 36028797018963967
72057594037927935 144115188075855871 288230376151711743 576460752303423487 1152921504606846975
2305843009213693951 4611686018427387903 9223372036854775807 18446744073709551615)
```

This is a list of $\mathrm{T}(1)$ through $\mathrm{T}(64)$ (the session was edited to put in line breaks for readability).